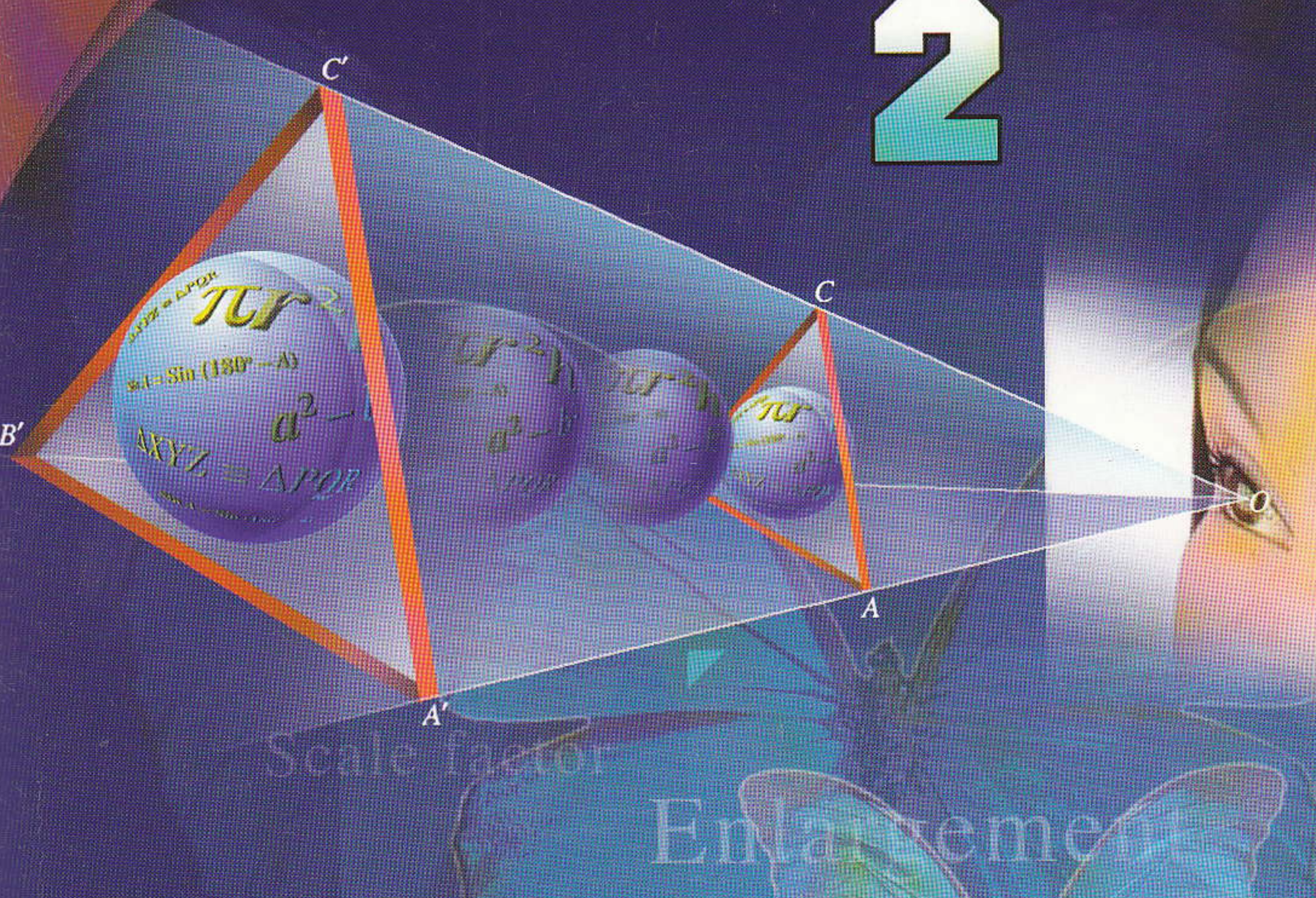
shinglee

New Syllabus

MATHEMATICS

6th Edition

2



Consultant:
Dr Yeap Ban Har

Authors:
Teh Keng Seng BSc, Dip Ed • **Loh Cheng Yee** BSc, Dip Ed
Joseph Yeo MEd, PGDE (Distinction), BSc (Hons) • **Ivy Chow** MEd, PGDE, BSc

CONTENTS

In this chapter, you will learn how to

- *identify congruent and similar figures;*
- *draw simple scale drawings with appropriate scales;*
- *solve problems involving congruence and similarity.*

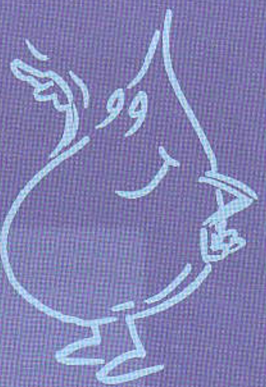
CHAPTER

1

Congruence and Similarity

Introduction

Look at the picture on the left. It shows flats of the same shape and size. This is an example of congruence. You can find many such examples from the environment around you.



Congruent Figures and Objects

Look at the picture of the two cars in Fig. 1.1. Is there any difference between them?

Fig. 1.1

The two cars have different colours and their positions in this textbook are different. But they have exactly the same shape and size. The two cars are said to be congruent. If you move the pictures of the two cars together, they will overlap exactly.

Two figures or objects are congruent if they have exactly the same shape and size. They need not be identical: they can have different colours or textures.

The two quadrilaterals shown in Fig. 1.2 have equal areas but not the same shape. They are thus not congruent.

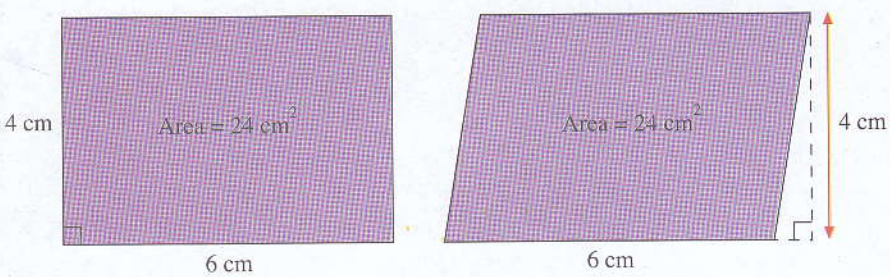


Fig. 1.2

Just for fun

Can you draw two straight lines through a square to divide it into four congruent quadrilaterals which are not parallelograms of any kind? Note that squares and rec-tangles are parallelograms.

The two regular hexagons shown in Fig. 1.3 have the same shape. However, they do not have the same size. They are thus not congruent. We say that the two hexagons are **similar**.

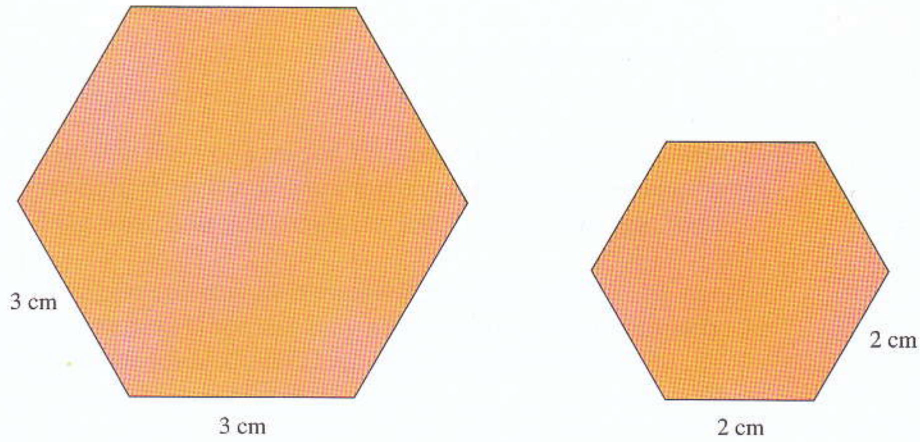


Fig. 1.3

Explorations

The concept of congruence is not restricted to the study of geometry. It plays an important part in everyday living too. We may be able to buy a refill for our pen when the ink runs dry. We are able to replace a worn-out part of a car with a new one of "the same part number". The blocks used in the construction of building toys are standard in size. The medium-sized diapers of a certain brand are all "alike". In each of these examples, the idea of "sameness of size and shape" comes in, and mathematically, we call this notion congruence.

Can you identify any congruent figures or objects in your classroom or school? Make a list of these congruent figures or objects by drawing or taking photos of them. Then share your findings with your classmates.

Example 1

Which shapes in Fig. 1.4 are congruent? Why?

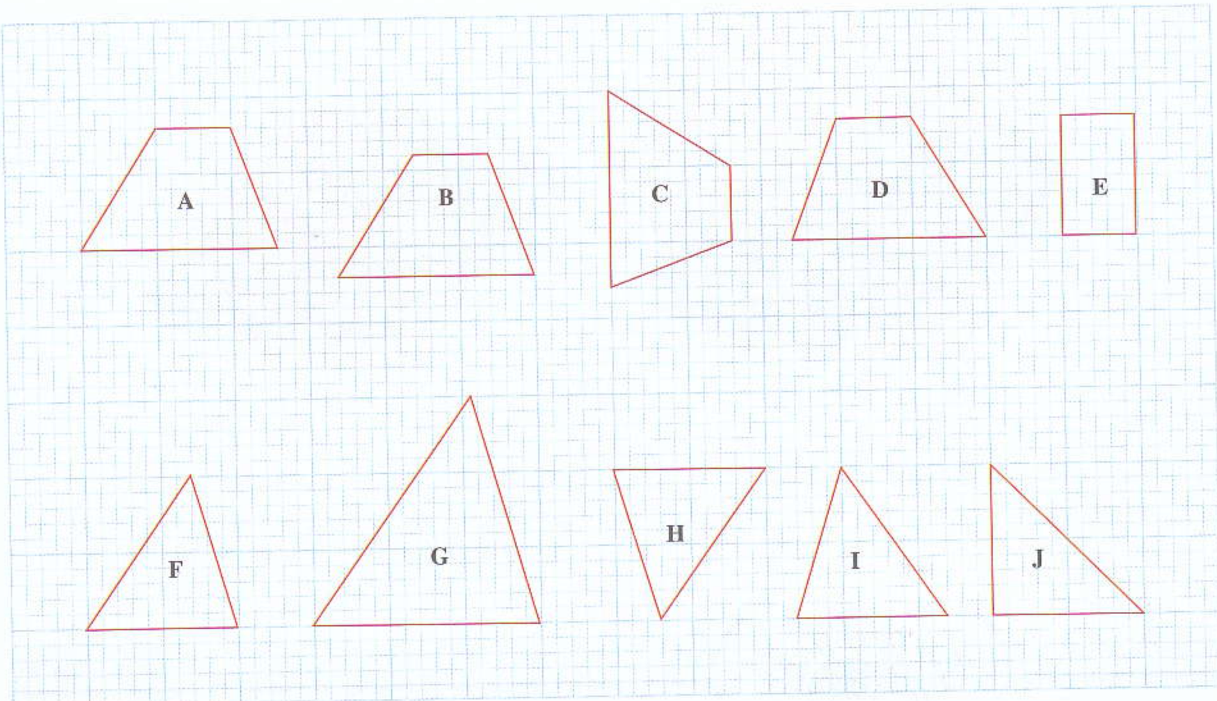


Fig. 1.4

Solution

A, B, C and **D** are congruent because they have the same shape and the same size.

In fact **B** can be obtained from **A** by **translation**, i.e. by moving **A** to the right 15 units and down 2 units.

C can be obtained from **A** by a **rotation**.

D can be obtained from **A** by a **reflection**.

E does not have the same shape as any of **A** – **J**, so it is not congruent to the others.

F, H and **I** are congruent because they have the same shape and the same size.

G has the same shape as **F** but it has a different size, so **F** and **G** are not congruent. In fact, **F** and **G** are similar.

J does not have the same shape as the other triangles, so it is not congruent to any of the other triangles.

From the above example, we know that translating, rotating and reflecting a shape or an object will result in an image that is congruent to the original shape or object.

Thinking Time

Place your left shoe in front of a mirror. Its image under reflection is exactly the same as your right shoe. But do your left and right shoes have exactly the same shape? If so, why can't you wear your left shoe on your right foot?

For Your Information

Although F to J in Fig. 1.4 are all triangular in shape, not all triangles have the same shape.

Fig. 1.5 shows two congruent quadilaterals $ABCD$ and $A'B'C'D'$. The vertex A corresponds to the vertex A' because they have the same angle. Similarly, the corresponding vertices for B, C and D are B', C' and D' respectively.

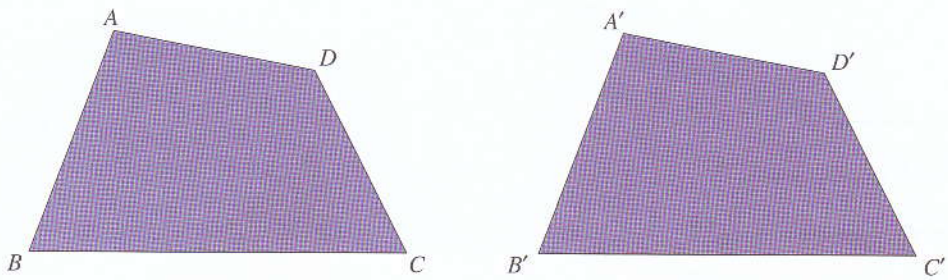


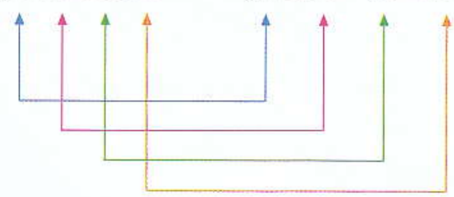
Fig. 1.5

The symbol ≡ means "is congruent to". So, in Fig. 1.5,

$$ABCD \equiv A'B'C'D'.$$

Notice that the vertices of $A'B'C'D'$ must be written in the same corresponding order of the vertices of $ABCD$:

$$ABCD \equiv A'B'C'D'$$



It is all right to write $BCDA \equiv B'C'D'A'$ because the corresponding vertices still match. Is it all right to write $CDAB \equiv C'D'A'B'$ or $DACB = D'A'B'C'$?

In Fig. 1.5, the side AD corresponds to the side $A'D'$. Similarly, the corresponding sides for AB, BC and CD are $A'B'$, $B'C'$ and $C'D'$ respectively.

It is obvious that the corresponding sides and angles of congruent figures are equal.

Example 2

Given that ABCD ≡ WXYZ, complete the following:

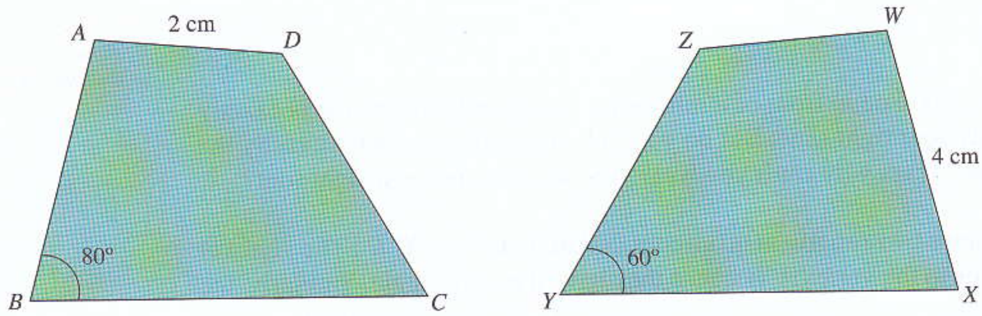


Fig. 1.6

(a) $A\hat{B}C = W\hat{X}Y =$ ________ ^{o}

(b) ________ $= X\hat{Y}Z =$ ________ ^{o}

(c) $AD =$ ________ $=$ ________ *cm*

(d) ________ $= WX =$ ________ *cm*

Solution

It may be helpful to write the corresponding vertices in this way:

$A \leftrightarrow W$
$B \leftrightarrow X$
$C \leftrightarrow Y$
$D \leftrightarrow Z$

(a) $A\hat{B}C = W\hat{X}Y = 80^\circ$

(b) $B\hat{C}D = X\hat{Y}Z = 60^\circ$ $(X \leftrightarrow B, Y \leftrightarrow C, Z \leftrightarrow D)$

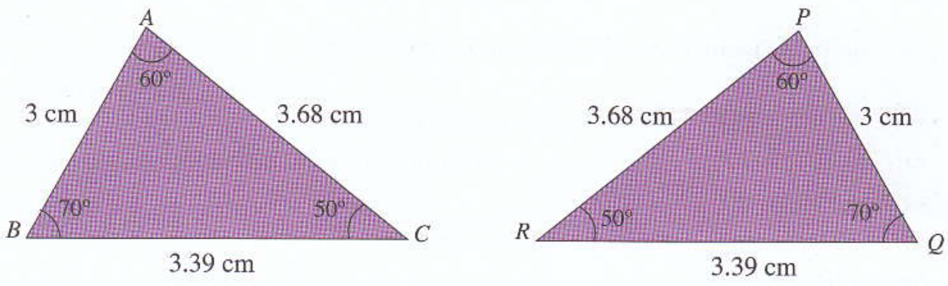
(c) $AD = WZ = 2$ cm $(A \leftrightarrow W, D \leftrightarrow Z)$

(d) $AB = WX = 4$ cm $(W \leftrightarrow A, X \leftrightarrow B)$

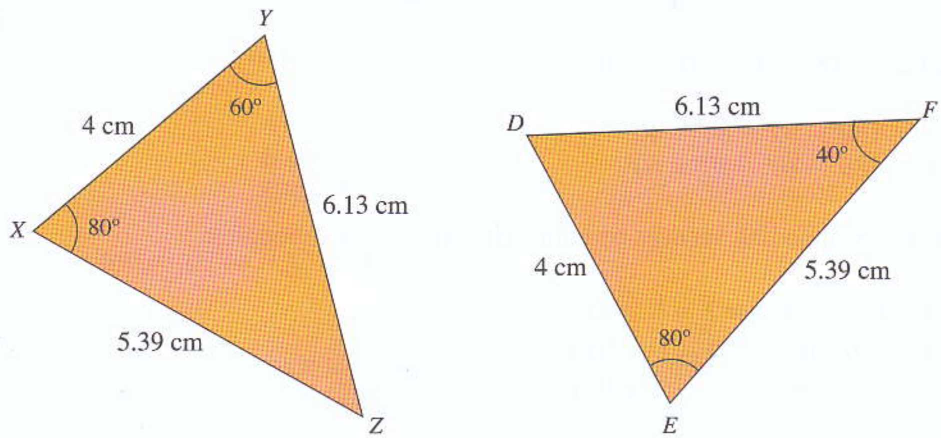
Example 3

Are the following pairs of figures congruent to each other? If so, explain why and write down the statement of congruence. If not, explain why.

(a)



(b)



(c)

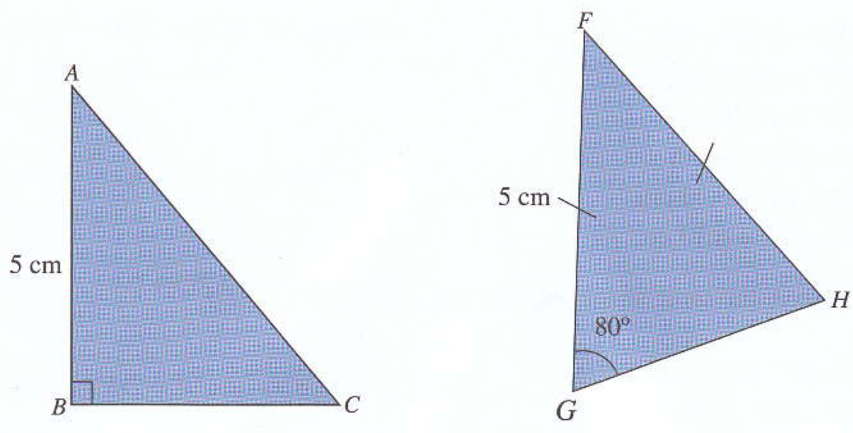


Fig. 1.7

Solution

(a) First, identify the corresponding vertices by comparing the size of the angles:

$$A \leftrightarrow P \quad (\text{because } \hat{A} = \hat{P} = 60^\circ)$$
$$B \leftrightarrow Q \quad (\text{because } \hat{B} = \hat{Q} = 70^\circ)$$
$$C \leftrightarrow R \quad (\text{because } \hat{C} = \hat{R} = 50^\circ)$$

Then, write proper statements using the above corresponding vertices:

$B\hat{A}C = Q\hat{P}R = 60^\circ$ (Notice the vertices are in order)

$A\hat{B}C = P\hat{Q}R = 70^\circ$

$A\hat{C}B = P\hat{R}Q = 50^\circ$

$AB = PQ = 3$ cm (Notice the vertices are in order)

$BC = QR = 3.39$ cm

$AC = PR = 3.68$ cm

$\therefore$ the two triangles have the same shape and size and so $\triangle ABC \equiv \triangle PQR$.

(b) First, find the last angle in each of the triangles:

$$\hat{Z} = 180^\circ - 60^\circ - 80^\circ = 40^\circ$$
$$\hat{D} = 180^\circ - 80^\circ - 40^\circ = 60^\circ$$

Then, identify two corresponding vertices by comparing the size of the angles:

$$X \leftrightarrow E \quad (\text{because } \hat{X} = \hat{E} = 80^\circ)$$
$$Y \leftrightarrow D \quad (\text{because } \hat{Y} = \hat{D} = 60^\circ)$$
$$Z \leftrightarrow F \quad (\text{because } \hat{Z} = \hat{F} = 40^\circ)$$

Then, we write proper statements using the above corresponding vertices:

$Y\hat{X}Z = D\hat{E}F = 80^\circ$

$X\hat{Y}Z = E\hat{D}F = 60^\circ$

$X\hat{Z}Y = E\hat{F}D = 40^\circ$

$XY = ED = 4$ cm

$XZ = EF = 5.39$ cm

$YZ = DF = 6.13$ cm

$\therefore$ the two triangles have the same shape and size and so $\triangle XYZ \equiv \triangle EDF$.

(c) In $\triangle FGH$, $\hat{H} = \hat{G} = 80^\circ$ (base $\measuredangle$s of isos. $\triangle$)

$$\hat{F} = 180^\circ - 80^\circ - 80^\circ = 20^\circ.$$

$\therefore$ $\triangle FGH$ does not have any right angle to correspond to that in $\triangle ABC$.

$\therefore$ $\triangle FGH$ does not have the same shape as $\triangle ABC$ and so it is not congruent to $\triangle ABC$.

Example 4

In the diagram, $\triangle ABC \equiv \triangle CED$. Given that $B\hat{A}C = 20^o$, $C\hat{D}E = 60^o$, $AB = 8.8$ cm and $CD = 10$ cm, find

(a) $E\hat{C}D$,
(b) $E\hat{C}B$,
(c) $A\hat{B}C$,
(d) the length of AC,
(e) the length of AE.

What can you say about the lines AB and CD?

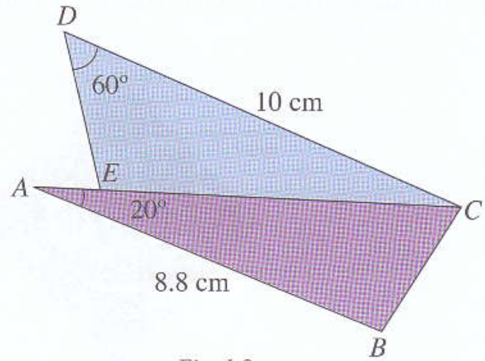


Fig. 1.8

Solution

Since $\triangle ABC \equiv \triangle CED$, it may be helpful to write the corresponding vertices in this way:

$$A \leftrightarrow C$$
$$B \leftrightarrow E$$
$$C \leftrightarrow D$$

(a) $E\hat{C}D = B\hat{A}C$ $(E \leftrightarrow B, C \leftrightarrow A, D \leftrightarrow C)$
$= 20^o$

(b) $E\hat{C}B = A\hat{C}B$ (This angle actually belongs to ABC: see the figure)
$= C\hat{D}E$ $(A \leftrightarrow C, C \leftrightarrow D, B \leftrightarrow E)$.
$= 60^o$

(c) $A\hat{B}C = 180^o - B\hat{A}C - A\hat{C}B$ ($\angle$ sum of $\triangle$)
$= 180^o - 20^o - 60^o$
$= 100^o$

(d) $AC = CD$ $(A \leftrightarrow C, C \leftrightarrow D)$
$= 10$ cm

(e) $EC = BA$ $(E \leftrightarrow B, C \leftrightarrow A)$
$= 8.8$ cm

$\therefore AE = AC - EC = 10 \text{ cm} - 8.8 \text{ cm} = 1.2 \text{ cm}$.

Since $B\hat{A}C = E\hat{C}D = 20^o$, then AB is parallel to CD. (alternate angles)

Explorations

Tessellations are beautiful patterns formed by congruent figures.

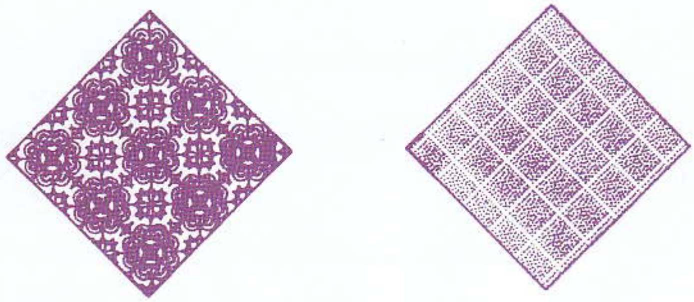
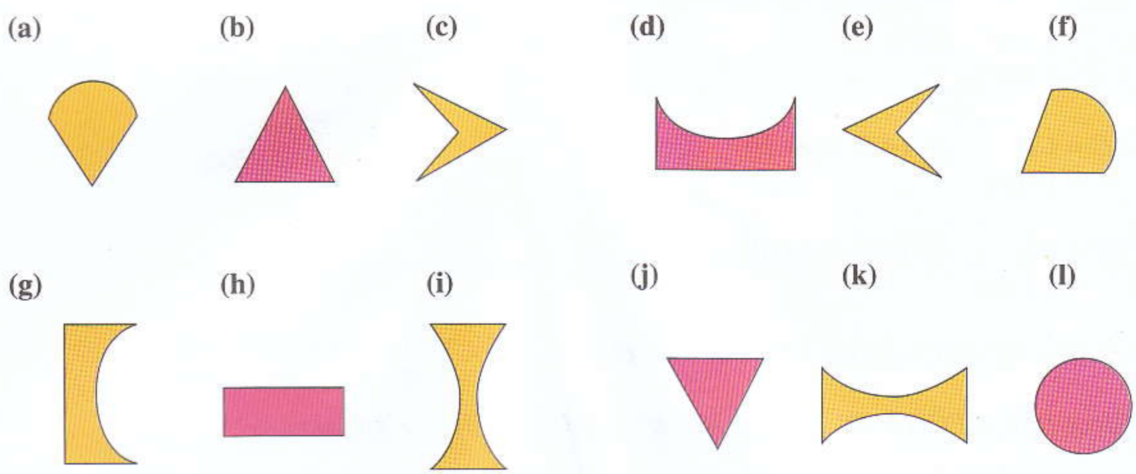


Fig. 1.9

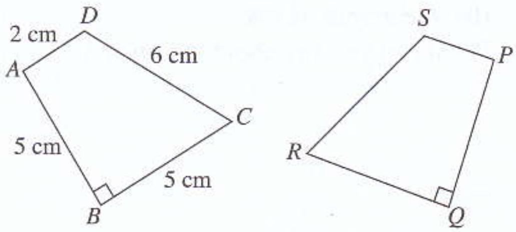
There are no gaps between the congruent figures. Look around you to see if there are any such tessellations, e.g. floor tiles, curtain prints and wall paper designs. You may want to search the internet to see art formed by tessellation of congruent figures. Try designing your own art using tessellation of congruent figures. You can colour it to make it more attractive! Then display your work on your class notice board.

Exercise 1a

1. Study the figures carefully and name the pairs that are congruent.



2. Given that $ABCD \equiv PQRS$, copy and complete the following:



(a) $PQ = AB =$ _______ cm

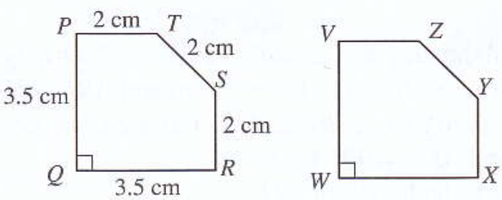
(b) $SR =$ _______ $= 6$ cm

(c) $PS =$ _______ $=$ _______ cm

(d) $QR =$ _______ $=$ _______ cm

(e) $P\hat{Q}R =$ _______ $=$ _______°

3. In the diagram below, $PQRST \equiv VWXYZ$.



Copy and complete the following:

(a) $PQ = VW =$ _______ cm

(b) $PT =$ _______ $= 2$ cm

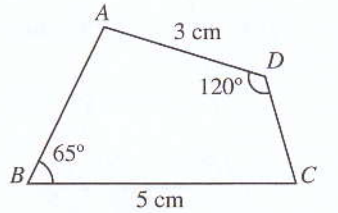
(c) $QR =$ _______ $=$ _______ cm

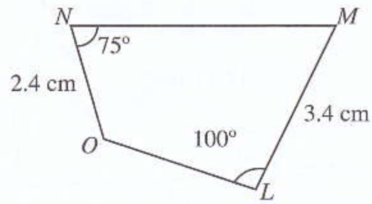
(d) $TS =$ _______ $=$ _______ cm

(e) $SR =$ _______ $=$ _______ cm

(f) $P\hat{Q}R =$ _______ $=$ _______°

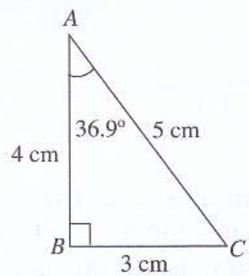
4. In the diagram below, $ABCD \equiv LMNO$. Write down the missing measurements.

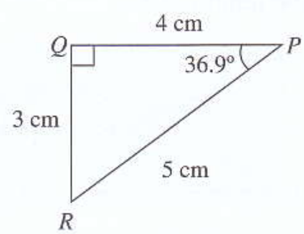




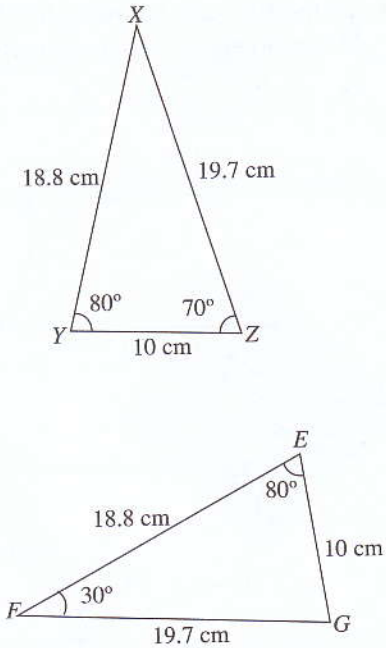
5. Are the following pairs of figures congruent to each other? If so, explain why and write down the statement of congruence. If not, explain why.

(a)

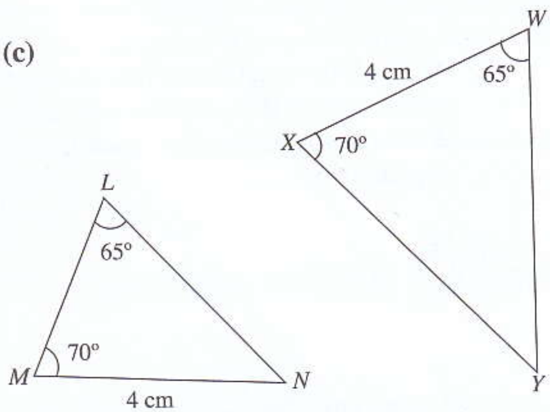




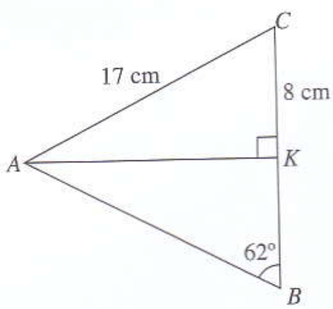
(b)



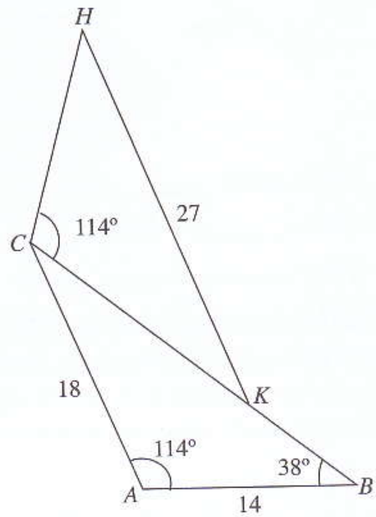
(c)



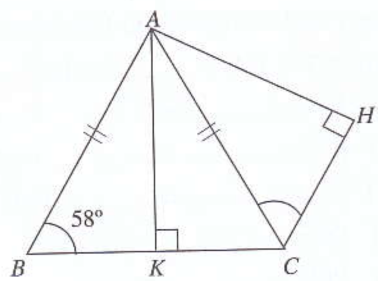
6. In the diagram, $\triangle ABK$ and $\triangle ACK$ are congruent. Given that AC = 17 cm, CK = 8 cm, $A\hat{K}C = 90°$ and $A\hat{B}K = 62°$, find
   (a) $B\hat{A}C$,
   (b) the length of BC.



7. In the diagram, $\triangle ABC$ and $\triangle CHK$ are congruent. Given that AB = 14 cm, AC = 18 cm, HK = 27 cm, $B\hat{A}C = H\hat{C}K = 114°$ and $A\hat{B}C = 38°$, find
   (a) $C\hat{K}H$,
   (b) the length of KB.
   What can you say about the lines AC and HK?



8. In the diagram, $\triangle ABC$ is an isosceles triangle where $AB = AC$, BC = 12 cm and $A\hat{B}K = 58°$. Given that $\triangle ABK$ and $\triangle ACH$ are congruent and $A\hat{K}C = 90°$, find
   (a) the length of CH,
   (b) $B\hat{A}H$.



Similar Figures and Objects

Fig. 1.10 (a) shows three mugs which look alike but are of different sizes.

Fig. 1.10 (b) shows three photographs which look alike but are of different sizes.

Fig. 1.10 (a)

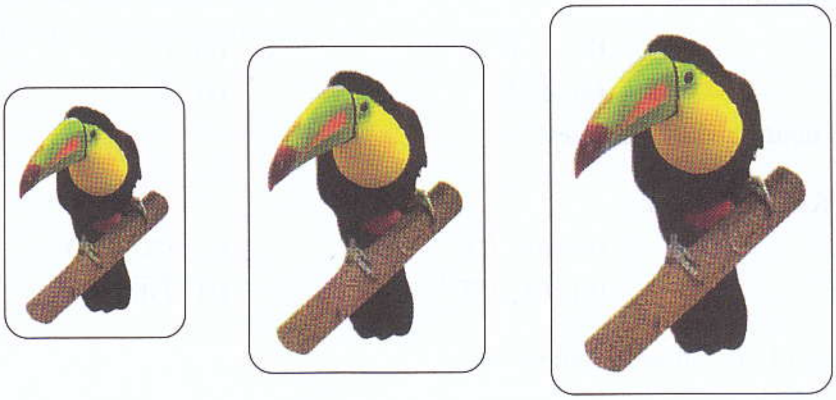


Fig. 1.10 (b)

Figures or objects that have the same shape but not necessarily the same size are said to be similar. When we think of the concept of similarity, many everyday situations come to our mind. For example, a road map, the plan of a house, a photograph, a magnifying glass, an overhead projector, a telescope, a microscope and a toy plane that is a scale model of a full-sized plane are all illustrations of the concept of similarity.

Two figures or objects are similar if they have exactly the same shape but not necessarily the same size. If they have exactly the same shape and size, then they are congruent.

Congruence is a special case of similarity.

Explorations

The hexagon in Fig. 1.11(a) increases in size without changing its shape.

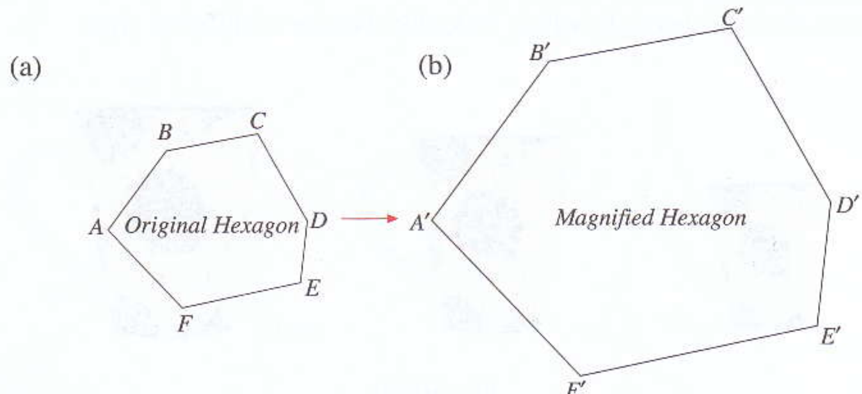


Fig. 1.11

1. Measure the following angles:

(a) $\hat{A}, \hat{A}'$ (b) $\hat{B}, \hat{B}'$ (c) $\hat{C}, \hat{C}'$

(d) $\hat{D}, \hat{D}'$ (e) $\hat{E}, \hat{E}'$ (f) $\hat{F}, \hat{F}'$

What do you notice about their sizes?

2. Measure the following sides:

(a) AB, $A'B'$ (b) BC, $B'C'$ (c) CD, $C'D'$

(d) DE, $D'E'$ (e) EF, $E'F'$ (f) FA, $F'A'$

3. Find the values of the following ratios:

(a) $\frac{A'B'}{AB}$ (b) $\frac{B'C'}{BC}$ (c) $\frac{C'D'}{AB}$ (d) $\frac{D'E'}{DE}$ (e) $\frac{E'F'}{EF}$ (f) $\frac{F'A'}{FA}$

What do you notice about their values?

From the above Exploration, we can conclude that if two figures are similar then the corresponding angles are the same and the length of each side is increased by the same factor, i.e.,

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'D'}{CD} = \frac{D'E'}{DE} = \frac{E'F'}{EF} = \frac{F'A'}{FA} = k, \text{ where } k \text{ is a constant.}$$

When two polygons are similar, then
1. **all the corresponding angles are equal, and**
2. **all the corresponding sides are proportional (or all the ratios of the corresponding sides are equal).**

Thinking Time

(a) Fig. 1.12 shows two rectangles.

(i) Are all the corresponding angles equal?

(ii) Are all the ratios of the corresponding sides equal?

(iii) Are the two rectangles similar?

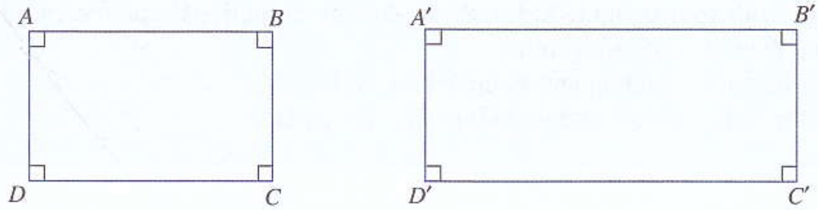


Fig. 1.12

(b) Fig. 1.13 shows a square and a rhombus.

(i) Measure the length of the square and the length of the rhombus. Are all the ratios of the corresponding sides equal?

(ii) Are all the corresponding angles equal?

(iii) Are the two quadrilaterals similar?

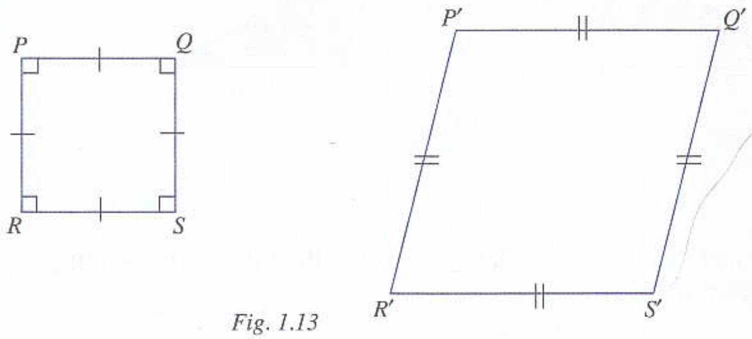


Fig. 1.13

(c) Fig. 1.14 shows two triangles.

(i) Measure all the angles. Are all the corresponding angles equal?

(ii) Measure the length of all the sides. Are all the ratios of the corresponding sides equal?

(iii) Are the two triangles similar?

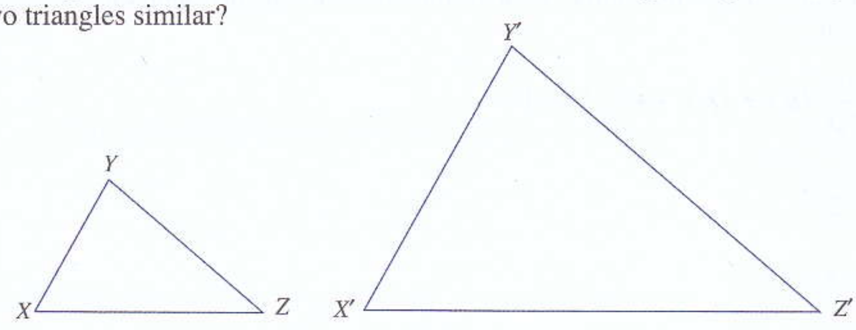


Fig. 1.14

(iv) Can you find two triangles which have equal corresponding angles but still not similar?

(v) Can you find two triangles which have equal ratios of corresponding sides but still not similar?

In conclusion,

(1) if two triangles have equal corresponding angles or equal ratios of corresponding sides, they are similar;

(2) if two polygons with four or more sides have equal corresponding angles or equal ratios of corresponding sides, they may or may not be similar.

For polygons with four or more sides, the following conditions must be true before we can conclude that the polygons are similar.

(i) All the corresponding angles are equal, and

(ii) all the ratios of the corresponding sides are equal.

Example 5

Given that $\triangle ABC$ is similar to $\triangle PQR$, find the unknowns in the figure.

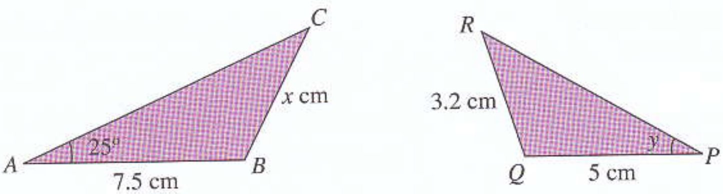


Fig. 1.15

Solution

Since it is given that "$\triangle ABC$ is similar to $\triangle PQR$", then the corresponding vertices or angles are in order:

$$A \leftrightarrow P$$
$$B \leftrightarrow Q$$
$$C \leftrightarrow R$$

Since $\triangle ABC$ and $\triangle PQR$ are similar, then all the ratios of the corresponding sides are equal.

$$\therefore \frac{BC}{QR} = \frac{AB}{PQ} \ (B \leftrightarrow Q, C \leftrightarrow R, A \leftrightarrow P)$$

i.e. $\frac{x}{3.2} = \frac{7.5}{5}$

$$\therefore x = \frac{7.5}{5} \times 3.2 = 4.8.$$

Since $\triangle ABC$ and $\triangle PQR$ are similar, then all the corresponding angles are equal.

$$\therefore y = Q\hat{P}R = B\hat{A}C \ (Q \leftrightarrow B, P \leftrightarrow A, R \leftrightarrow C)$$
$$= 25^\circ$$

Notes

*There is **no standard notation for similarity**. Do not use ≅ for similarity because some countries use this symbol for congruence! So we just write "$\triangle ABC$ and $\triangle PQR$ are similar."*

Example 6

Quadrilateral $ABCD$ is similar to quadrilateral $PQRS$. Find the unknowns in the figure.

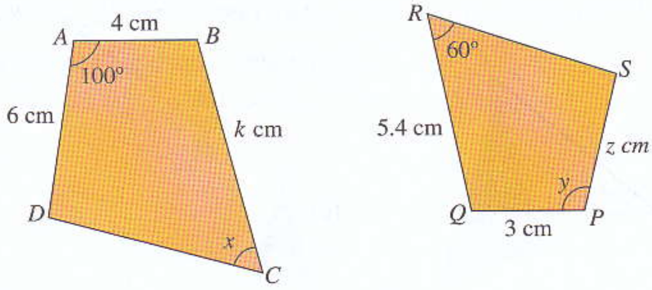


Fig. 1.16

Solution

Since $ABCD$ and $PQRS$ are similar, then $A \leftrightarrow P$
$B \leftrightarrow Q$
$C \leftrightarrow R$
$D \leftrightarrow S$

$\therefore x = B\hat{C}D = Q\hat{R}S$ (corr. angles are equal)
$= 60°$

$\therefore y = Q\hat{P}S = B\hat{A}D$ (corr. angles are equal)
$= 100°$

$\dfrac{BC}{QR} = \dfrac{AB}{PQ}$ (ratios of corr. sides are equal)

i.e. $\dfrac{k}{5.4} = \dfrac{4}{3}$

$\therefore k = \dfrac{4}{3} \times 5.4 = 7.2$

$\dfrac{PS}{AD} = \dfrac{PQ}{AB}$ (ratios of corr. sides are equal)

ie. $\dfrac{z}{6} = \dfrac{3}{4}$

$\therefore z = \dfrac{3}{4} \times 6 = 4.5$

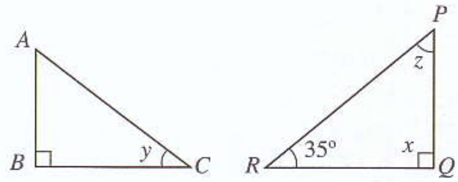
Problem Solving Tips

It may be helpful to write the unknown side BC on the top left hand corner. Then we match $B \leftrightarrow Q$ and $C \leftrightarrow R$. We use $\frac{AB}{PQ}$ because we know the lengths of AB and PQ.

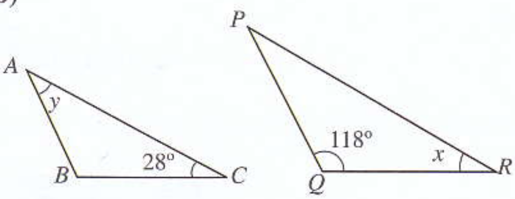
Exercise 1b

1. In each of the following triangles, $\triangle ABC$ is similar to $\triangle PQR$. Find the unknowns in the figures.

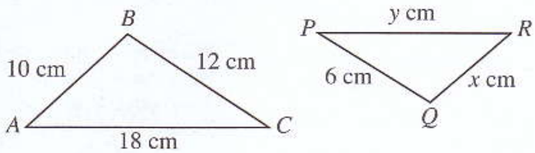
(a)



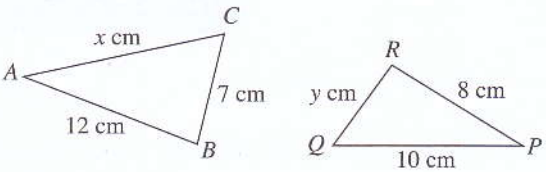
(b)



(c)

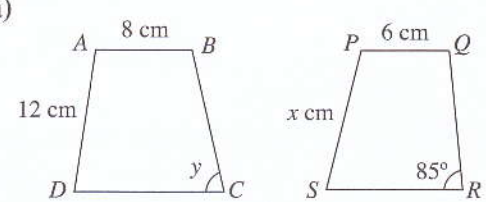


(d)

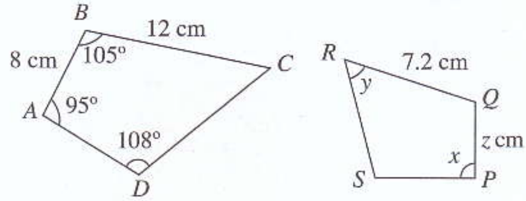


2. In each of the following cases, quadrilateral $ABCD$ is similar to quadrilateral $PQRS$. Find the unknowns in the figures.

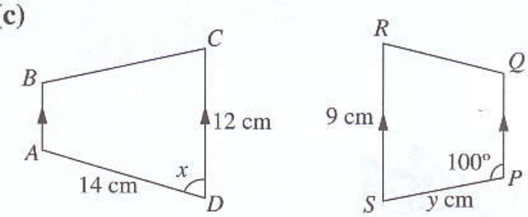
(a)



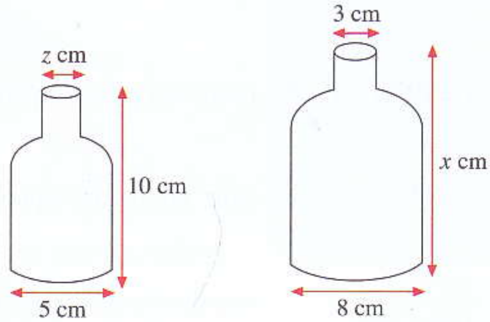
(b)



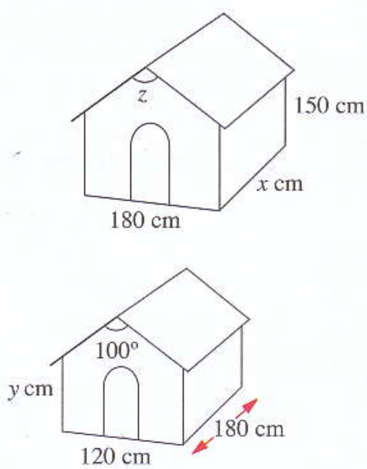
(c)



3. Calculate the unknown lengths of the two similar water bottles shown below.



4. Calculate the unknowns marked in the two similar toy houses shown below.



Similarity and Enlargement

Fig. 1.17 shows two similar triangles ABC and $A'B'C'$. $\triangle A'B'C'$ is an enlargement of $\triangle ABC$. We say that $\triangle ABC$ is mapped onto $\triangle A'B'C'$ by an enlargement with centre O and scale factor $\frac{OA'}{OA}$.

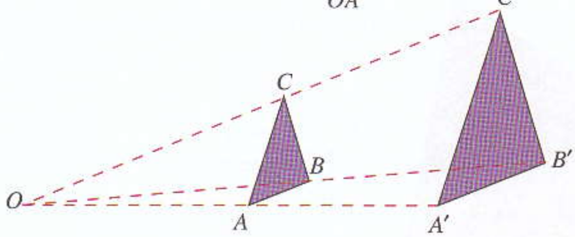
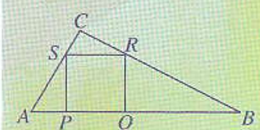


Fig. 1.17

There are many examples of enlargement in our surroundings. An example is the enlargement of a photograph. Also, when your teacher uses an overhead projector to project a picture on a transparency, he is enlarging the picture. Are the picture and its projected image similar?

Just for fun

Draw a triangle ABC such that AB = 8 cm, BC = 6 cm and AC = 4 cm. By using enlargements, construct a square PQRS inside the triangle such that PQ is on AB, R on BC and S on AC.



Explorations

Use Fig. 1.17 to answer the following questions.

(a) Find the ratios , $\frac{OA'}{OA}$, $\frac{OB'}{OB}$ and $\frac{OC'}{OC}$ by measuring the appropriate lengths.

What do you notice? What is the scale factor of the enlargement?

(b) Find the ratios of corresponding sides $\frac{A'B'}{AB}$, $\frac{B'C'}{BC}$ and $\frac{A'C'}{AC}$ by measuring the appropriate lengths.

(i) What do you notice?

(ii) What can you say about the ratios of the corresponding sides and the scale factor of the enlargement?

(iii) What can you say about $\triangle ABC$ and its enlarged image $\triangle A'B'C'$? Why?

(c) What do you think will happen if $\triangle ABC$ undergoes an enlargement with centre O and scale factor $\frac{1}{2}$?

(d) What do you think will happen if $\triangle ABC$ undergoes an enlargement with centre O and scale factor 1?

Notes

An enlargement with a scale factor between 0 and 1 will result in a smaller image. But it is still called an enlargement in mathematics.

Example 7

In Fig. 1.18, $\triangle PQR$ is mapped onto $\triangle PQ'R'$ by an enlargement with centre P and scale factor 2. If PQ = 4 cm, PR = 5 cm, find the length of PQ′ and PR′.

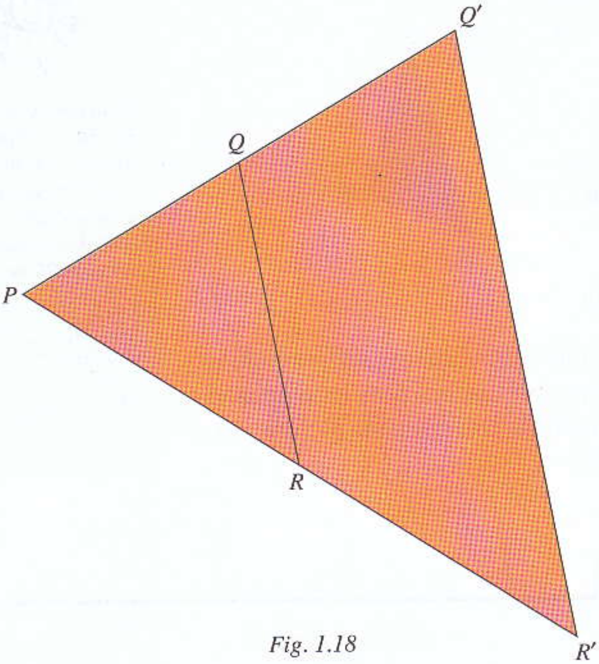
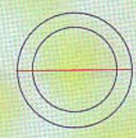
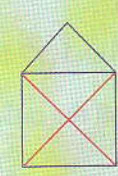
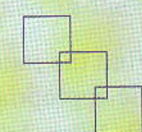


Fig. 1.18

> **Just for fun**
>
> *Draw each figure without lifting your pen from the paper or retracting any line.*
>
> *1.*
>
> 
>
> *2.*
>
> 
>
> *3.*
>
> 

Solution

$\triangle PQR$ is similar to $\triangle PQ'R'$ under enlargement.

$\therefore \quad \dfrac{PQ'}{PQ} = \dfrac{PR'}{PR} = 2$ (scale factor)

i.e. $\dfrac{PQ'}{4} = 2$

and $\dfrac{PR'}{5} = 2$

$\therefore \quad PQ' = 8$ cm

and $PR' = 10$ cm

Similarity and Scale Drawings

In our daily activities, we sometimes need to enlarge or reduce pictures or drawings of actual objects. For example, if we wish to draw a plan of a badminton court in order to explain the rules of the game, we need to make our drawing very much smaller on paper or on a whiteboard. If we wish to show a diagram of the apparatus used for the preparation of oxygen, we can enlarge the diagram with the use of an overhead projector.

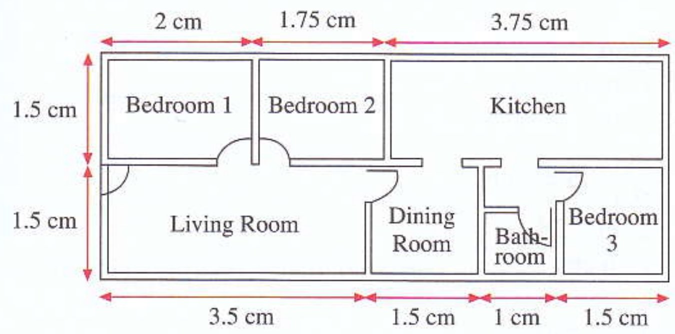


Fig. 1.19

Fig. 1.19 shows the floor plan of a terrace house. It is similar to the actual floor of the house. The dimensions of the plan are proportional to the corresponding actual dimensions of the house. Fig. 1.19 has been drawn to a scale of 1 cm to 2 m, i.e., 1 cm on the plan represents 2 m on actual ground. From Fig. 1.19, we find the following:

(a) The length of the living room is 3.5×2 m $= 7$ m and its width is 1.5×2 m $= 3$ m.

(b) The area of bedroom 1 is 1.5 cm $\times$ 2 cm on the plan.
Hence, actual area $= (1.5 \times 2)$ m $\times (2 \times 2)$ m $= 12$ m^2.

(c) The area of the living room $= (1.5 \times 2)$ m $\times (3.5 \times 2)$ m $= 21$ m^2.

(d) The total area of the house $= (3 \times 2)$ m $\times (7.5 \times 2)$ m $= 90$ m^2.

Example 8

The scale of a building plan is 1 cm to 50 cm. Find (a) the actual length of one of the bedrooms if it is represented by a length of 9.2 cm and (b) the length on the plan that represents an actual length of 28 m.

Solution

(a) **Plan** → **Actual**

1 cm → 50 cm (scale)

9.2 cm → 50 cm × 9.2

= 460 cm

= 4.6 m

∴ the actual length is 4.6 m.

(b) **Actual** → **Plan**

50 cm → 1 cm (scale)

i.e. 0.5 m → 1 cm

1 m → $\frac{1}{0.5}$ cm = 2 cm

28 m → 2 cm × 28

= 56 cm

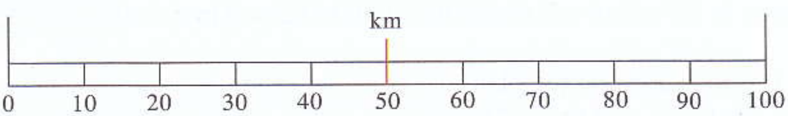
∴ the length on the plan is 56 cm.

Notes

Always write what you want to find on the right. In this case, you want to find the length on the plan. So write 'Plan' on the right.

Scales on Maps

Maps are scale drawings of actual land. The scale of a map is usually given at a corner of the map. There are several ways of representing the scale of a map. For example, on a map of Singapore, the following scale may be given:



Scale 1 : 1 000 000

There are two ways to read the scale. Use your ruler to measure the length from 0 to 10 km and you will find that it is actually 1 cm. So 1 cm represents 10 km. This is the same as the scale 1 : 1 000 000. When a scale is given in this form, it means you must use the same units on both sides, i.e. 1 m : 1 000 000 m, or 1 km : 1 000 000 km, or 1 cm : 1 000 000 cm = 10 km, so 1 cm represents 10 km.

The scale of 1 : 1 000 000 can also be expressed as a representative fraction (R.F.) of $\frac{1}{1\,000\,000}$. For example, if the R.F. is $\frac{1}{200}$, the scale will be 1 : 200. When we use R.F., the numerator must always be 1.

Example 9

A map has a scale of 1 cm to 3 km.

(a) What length on actual round does a 3-cm length on the map represent?
(b) What length will represent 7.5 km on the map?
(c) What is the R. F. of the map?

Solution

(a) **Map** → **Actual**

1 cm → 3 km (scale)

3 cm → 3 km × 3
= 9 km

∴ a 3-cm length on the map represents 9 km on actual ground.

(b) **Actual** → **Map**

3 km → 1 cm (scale)

1 km → $\frac{1}{3}$ cm

7.5 km → $\frac{1}{3}$ cm × 7.5 = 2.5 cm

∴ 2.5 cm will represent 7.5 km.

Notes

*Always write what you want to find on the right. In this case, you want to find the length on the **map**. So write 'map' on the right.*

(c) 3 km = 300 000 cm

i.e. the R.F. of the map is $\frac{1}{300\,000}$.

∴ the scale is 1 : 300 000.

Using the scale of a map, we can also find the actual area of a site from its area on the map. For example, if the scale of a map is 1 cm to 2 km, then 1 cm^2 on the map will represent $(2\text{ km})^2 = 4\text{ km}^2$ (see Fig. 1.20).

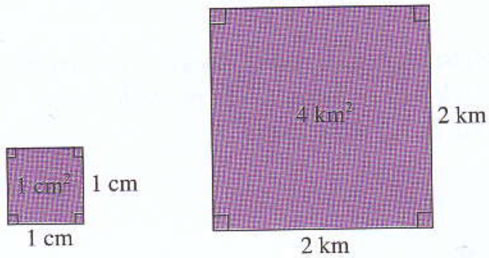


Fig. 1.20

Example 10

A scale of 2 cm to 1 km is used for a map.

(a) A piece of land has an area of 8 cm^2 on the map. What is its actual area?
(b) A pond has an actual area of 50 000 m^2. What is its area on the map?

Solution

The scale of the map is 2 cm to 1 km.
So $(2\text{ cm})^2 = 4\text{ cm}^2$ will represent $(1\text{ km})^2 = 1\text{ km}^2$.

(a) **Map** → **Actual**

$4\text{ cm}^2 \longrightarrow 1\text{ km}^2$

$8\text{ cm}^2 \longrightarrow 1\text{ km}^2 \times \frac{8}{4}$

$= 2\text{ km}^2$

$\therefore$ the actual area is 2 km^2.

(b) **Actual** → **Map**

$1\text{ km}^2 \longrightarrow 4\text{ cm}^2$

$1000\,000\text{ m}^2 \longrightarrow 4\text{ cm}^2$

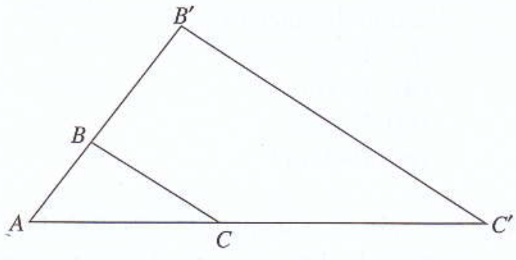
$50\,000\text{ m}^2 \longrightarrow 4\text{ cm}^2 \times \frac{50\,000}{1000\,000}$

$= 0.2\text{ cm}^2$

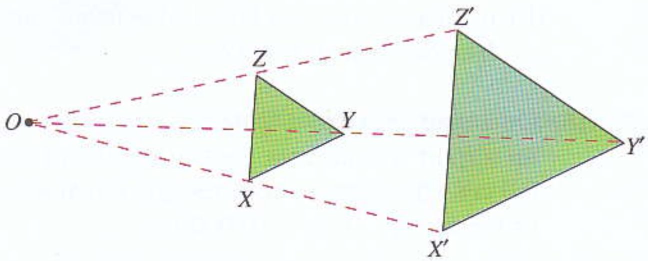
$\therefore$ the area of the pond on the map is 0.2 cm^2.

Exercise 1c

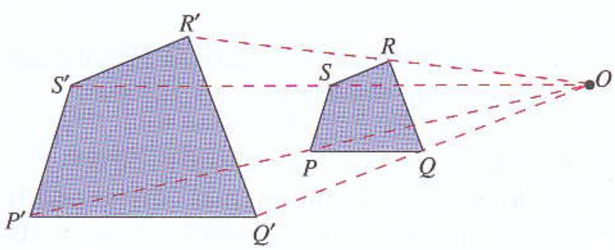
1. In the figure below, $\triangle ABC$ is mapped onto $\triangle AB'C'$ by an enlargement with centre A and scale factor 3. If $AB = 3$ cm and $B'C' = 12$ cm, find the length of AB' and BC.



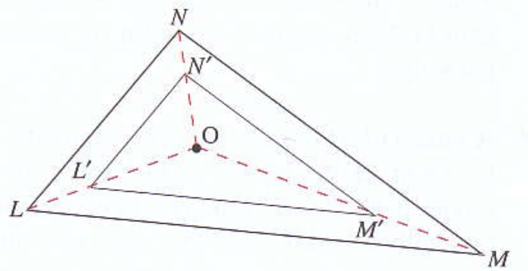
2. In the figure below, $\triangle XYZ$ is mapped onto $\triangle X'Y'Z'$ by an enlargement with centre O and scale factor 2.5. If $OX = 3$ cm, $XY = 4$ cm and $Y'Z' = 8.75$ cm, find the length of OX', $X'Y'$ and YZ.



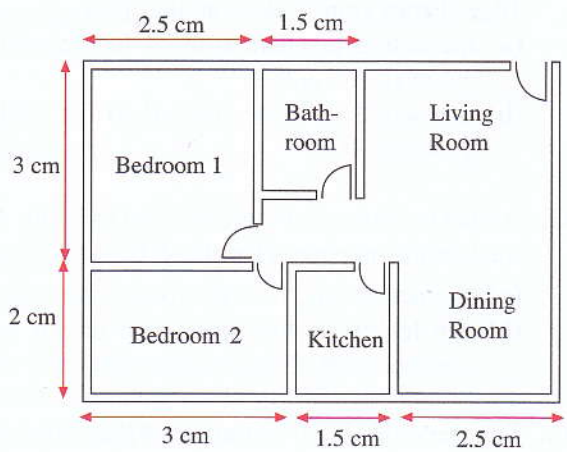
3. In the figure below, $\triangle PQRS$ is mapped onto $P'Q'R'S'$ by an enlargement with centre O and scale factor k.
   (a) If $OP = 8$ cm and $PP' = 8$ cm, find the scale factor k.
   (b) If $OQ = 4$ cm and $RR' = 5$ cm, find the length of OQ' and OR'.



4. In the figure below, $\triangle LMN$ is mapped onto $\triangle L'M'N'$ by an enlargement with centre O and scale factor $\frac{1}{2}$. If $OL = 6$ cm and $OM' = 3.5$ cm, find the lengths of OL' and OM.



5. The diagram below shows the floor plan of a flat.
   The scale is 1 cm to 1.5 m.
   Find the actual
   (a) dimensions and area of bedroom 1;
   (b) dimensions and area of the kitchen;
   (c) total area of the flat.



6. A scale model of a house is made to a scale of 1 cm to 3 m.
   (a) Given that the length of the model is 12 cm, calculate the actual length of the house.
   (b) If the actual width of the house is 12 m, calculate the width of the model.

7. On a scale drawing, the length of a ship is 45 cm. The actual length of the ship is 90 m. What is the scale used? If the width of the ship is 25 m, what is its width on the scale drawing?

8. A model of a tower is made to a scale of 1 cm to 0.5 m. If the height of the tower on the model is 84 cm, find the height of the tower on another model made to a scale of 1 cm to 2 m.

9. Make a scale drawing of a rectangular school hall which is 50 m long and 30 m wide using a scale of 1 cm to 5 m. Use your scale drawing to find the actual distance between the opposite corners of the hall.

10. A triangular field ABC is such that $AB = 90$ m, $BC = 70$ m and $AC = 85$ m. Make a scale drawing of it, using a scale of 1 cm to 10 m. From your scale drawing, find
    (a) the actual distance from A to the mid-point of BC and,
    (b) the actual distance from B to the mid-point of AC.

11. A map is drawn to a scale of 5 cm to 2 km. A road on the map has a length of 12.5 cm. Find
    (a) the actual length of the road in km,
    (b) the length of the same road drawn on another map of scale 3 cm to 10 km.

12. The scale of a map is 1 cm to 5 km. What is the distance on the map between two towns if they are actually
    (a) 25 km apart,
    (b) 38 km apart,
    (c) 12.5 km apart,
    (d) 2500 m apart?

13. A map is drawn to a scale of 1 : 50 000.
    (a) What is the actual distance represented by
        (i) 2 cm,
        (ii) 7.5 cm,
        (iii) 0.6 cm,
        (iv) 26 cm?
    (b) What length on the map represents the actual distance of
        (i) 4 km,
        (ii) 15 km,
        (iii) 250 m,
        (iv) 1600 m?

14. The scale of a map is 1 : 20 000. Find the distance in km of a road represented by $5\frac{1}{2}$ cm on the map?

15. Given that 1 cm on a map represents 2 km on the ground, find
    (a) the actual area of a plot of land if the area of the land on the map is 3 cm^2,
    (b) the actual area of a lake if the area of the lake on the map is 4.5 cm^2.

16. On a map drawn to a scale of 1 cm to 500 m, an airport has an area measuring 16 cm by 8.5 cm. Calculate, in hectares, the actual area of the airport. (1 ha = 10 000 m^2)

17. Given that 1 cm on a map represents 2 km on the ground, calculate the area of a park on the map if the actual area of the park is 10 km^2.

18. The scale of a map is 1 cm : 8 km. What area on the map would represent
    (a) 64 km^2,
    (b) 128 km^2,
    (c) 320 km^2,
    (d) 1600 km^2?

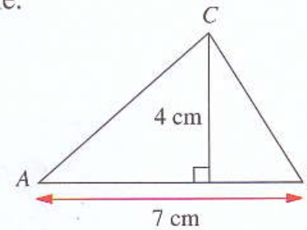
19. The scale of a map is 1 : 20 000. Find the area on the map which represents 124 km^2.

20. A map is drawn to a scale of 1 : 50 000.
    (a) Calculate the actual distance, in kilometres, represented by 4 cm on the map.

(b) Two towns are 28 km apart. Calculate, in centimetres, their distance apart on the map.

(c) On the map, a forest has an area of 12 cm^2. Calculate, in square kilometres, the actual area of the forest.

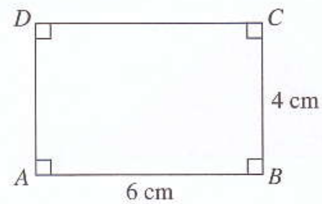
21. In the given figure, ABC is a triangle of height 4 cm and base 7 cm, drawn to a scale of 1 cm : 3 km. Find the actual area of the triangle.

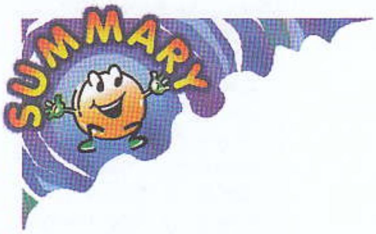


22. The figure below is a rectangle, drawn to a scale of 1 cm : 200 m. Given that $AB = 6$ cm and $BC = 4$ cm, find the actual

(a) length and breadth of the rectangle in km,

(b) area of the rectangle in km^2.





1. Congruent figures or objects have exactly the same shape and the same size.

2. A figure and its image under translation, rotation or reflection are congruent.

3. Similar figures or objects have exactly the same shape but not necessarily the same size.

4. Two polygons are similar if
   (a) all the corresponding angles are equal, and
   (b) all the ratios of the corresponding sides are equal.

5. Congruence is a special case of similarity.

6. A figure and its image under an enlargement are similar.

7. An enlargement with a scale factor greater than 1 produces an enlarged image.
   An enlargement with a scale factor between 0 and 1 produces a diminished image.
   An enlargement with a scale factor of 1 produces a congruent image.

8. If the linear scale of a map is $1 : x$, it means that 1 cm on the map represents x cm on the actual piece of land.

Review Examples 1

Example 1

(a)

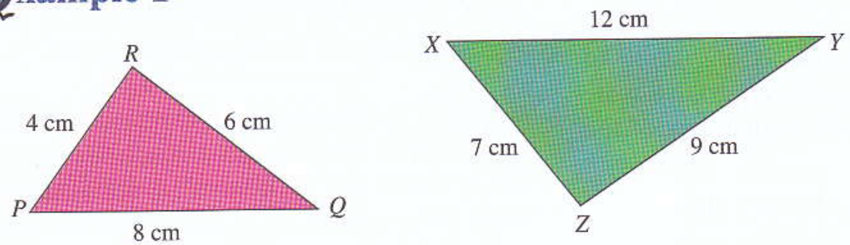


Fig. 1.21 (a)

Are △PQR and △XYZ similar?

(b)

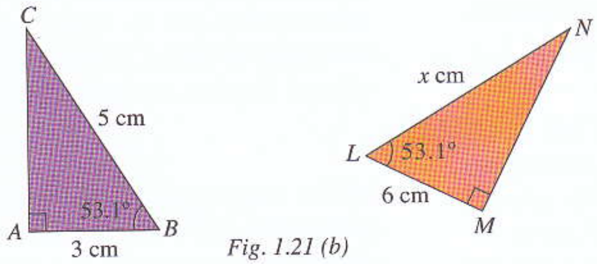


Fig. 1.21 (b)

△ABC and △LMN are similar. Find the value of x.

Solution

(a) $\dfrac{XY}{PQ} = \dfrac{12}{8} = 1.5$

$\dfrac{YZ}{QR} = \dfrac{9}{6} = 1.5$

$\dfrac{XZ}{PR} = \dfrac{7}{4} = 1.75$

Since not all the ratios of the corresponding sides are equal, then △*PQR* and △*XYZ* are not similar.

(b) △*ABC* and △*LMN* are similar. Sometimes, the vertices given may not be corresponding. So we need to check:

$A \leftrightarrow M$ (because $\hat{A} = \hat{M} = 90°$)

$B \leftrightarrow L$ (because $\hat{B} = \hat{L} = 53.1°$)

$C \leftrightarrow N$ (third angles must be equal).

$\therefore$ △*ABC* is similar to △*MLN* (we always write in order).

$\therefore \dfrac{LN}{BC} = \dfrac{LM}{BA}$ i.e, $\dfrac{x}{5} = \dfrac{6}{3}$

$\therefore x = \dfrac{6}{3} \times 5 = 10$

Problem Solving Tips

*How do you know which are corresponding sides? If △PQR and △XYZ are similar, then the **longest side** of △PQR will correspond to the longest side of △XYZ, the second longest side of △PQR will correspond to the second longest side of △XYZ, and the **shortest** side of △PQR will correspond to the **shortest** side of △XYZ.*

Thus, we need to compare the longest side of △PQR with the longest side of △XYZ, the second longest side of △PQR with the second longest side of △XYZ, and the shortest side of △PQR with the shortest side of △XYZ.

Example 2

(a) John makes a model plane using a scale of 1 : 20. The model plane has an overall length of 1.2 m and its wingspan is 65 cm. Find the wingspan and the overall length of the actual plane in metres.

(b) On another occasion, he makes a model of another plane using a scale of 1 : 25. What is the area of the tail section of the model in square centimetres if its actual area is 3 m^2?

Solution

(a)

Model		**Actual**
1 cm	→	20 cm
120 cm	→	$20 \text{ cm} \times 120 = 2400 \text{ cm} = 24 \text{ m}$
65 cm	→	$20 \text{ cm} \times 65 = 1300 \text{ cm} = 13 \text{ m}$

$\therefore$ the wingspan and the overall length of the actual plane are 24 m and 13 m respectively.

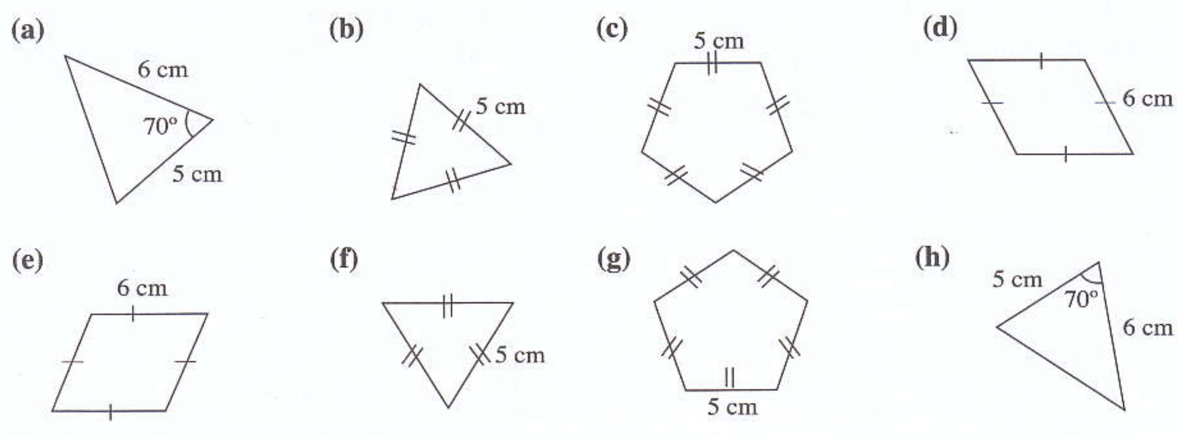
(b)

Actual		**Model**
25 m	→	1 m
1 m	→	$\frac{1}{25} \text{ m} = 4 \text{ cm}$
1 m^2	→	$(4 \text{ cm})^2 = 16 \text{ cm}^2$
3 m^2	→	$16 \text{ cm}^2 \times 3 = 48 \text{ cm}^2$

$\therefore$ the area of the tail section of the model is 48 cm^2.

Review Questions 1

1. Study the figures below carefully and identify the 4 pairs of congruent figures.



2. Given that $\triangle ABC$ is congruent to $\triangle PQR$, $\hat{A} = 70°$, $\hat{B} = 60°$ and $AB = 8$ cm, find

(a) $\hat{P}$,

(b) $\hat{Q}$,

(c) $\hat{R}$,

(d) PQ.

3. Given that $ABCD$ is congruent to $PQRS$, $\hat{A} = 100°$, $\hat{B} = 70°$, $\hat{C} = 95°$ and $PQ = 6$ cm, find

(a) $\hat{P}$,

(b) $\hat{Q}$,

(c) $\hat{R}$,

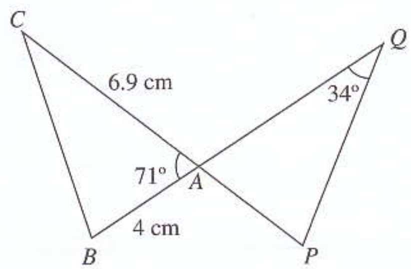
(d) $\hat{S}$,

(e) AB.

4. In the diagram, $\triangle APQ$ and $\triangle ABC$ are congruent. Given that $AB = 4$ cm, $AC = 6.9$ cm, $B\hat{A}C = 71°$ and $A\hat{Q}P = 34°$, find

(a) the length of BQ,

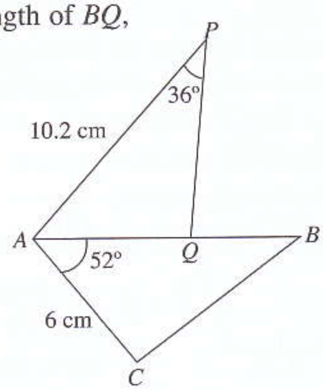
(b) ABC.



5. In the diagram, $\triangle ABC \equiv \triangle APQ$. Given that $B\hat{A}C = 52°$, $A\hat{P}Q = 36°$, $AC = 6$ cm and $AP = 10.2$ cm. Find

(a) the length of BQ,

(b) AQP.

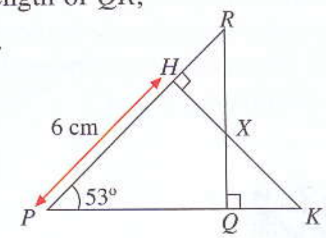


6. In the diagram, $\triangle PQR \equiv \triangle PHK$, $Q\hat{P}R = 53°$, the area of $\triangle PQR = 24$ cm^2, $PH = 6$ cm and $P\hat{Q}R = 90°$. Find

(a) the length of PQ,

(b) the length of QR,

(c) QXK.



7. Given that $\triangle ABC$ is similar to $\triangle PQR$, $\hat{A} = 50°$ and $\hat{B} = 68°$, find

(a) $\hat{P}$,

(b) $\hat{Q}$,

(c) $\hat{R}$.

8. Given that $\triangle ABC$ is similar to $\triangle PQR$, $AB = 6$ cm, $AC = 8$ cm, $\hat{A} = 60°$ and $PR = 10$ cm, find

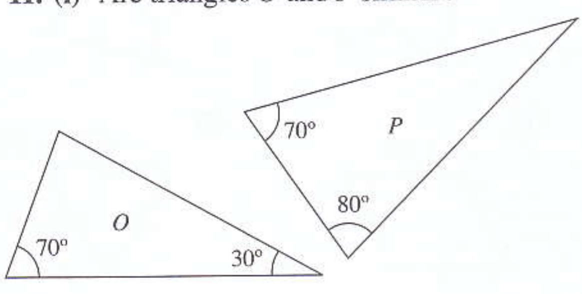
(a) $\hat{P}$ and

(b) PQ.

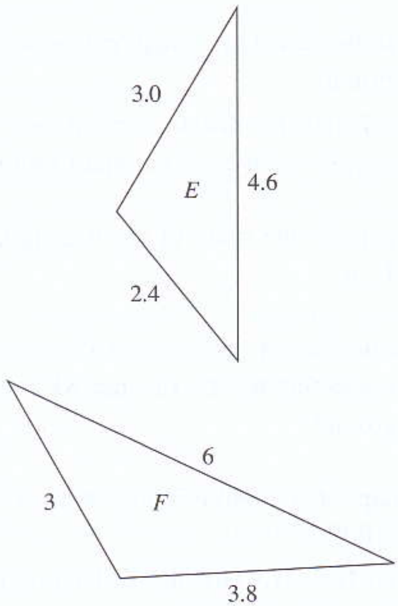
9. A television image of a teddy bear is 6 cm tall and that of its owner, a little girl, is 15 cm tall. If the actual height of the little girl is 120 cm, what is the height of her teddy bear?

10. A photograph is taken of a man 180 cm tall, standing in front of his terrace house. The image of the man in the photograph is 9 cm tall and that of his house is 22.5 cm tall. Calculate the actual height of the house.

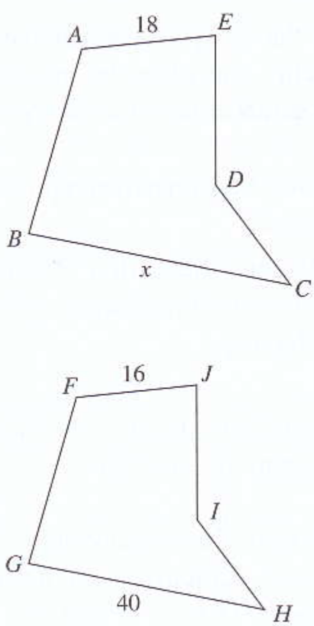
11. (i) Are triangles O and P similar?



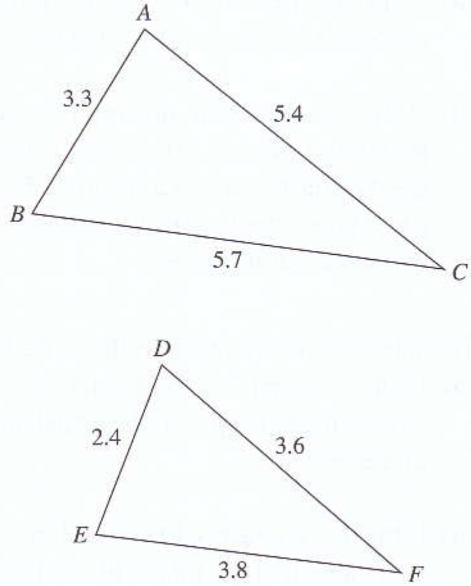
(ii) Are triangles E and F similar?



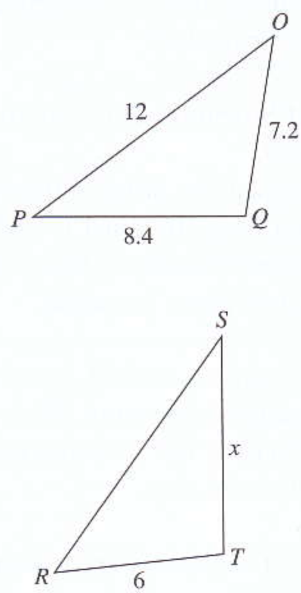
(iii) The shapes $ABCDE$ and $FGHIJ$ are similar. Calculate the value of x.



12. (a) Are triangles ABC and DEF similar?



(b) Triangles OPQ and RST are similar. Calculate the value of x.



13. A man is 1.75 m high and the height of a tree is 8.75 m.
   (a) Express the height of the man as a percentage of the height of the tree.
   (b) Both the tree and the man stand on horizontal ground. At a certain time of the day, the tree casts a shadow of length 6 m. What's the length of the shadow cast by the man at the time?

14. The model of an aircraft is in the scale 1: 80.
   (a) If the wingspan of the model is 25 cm, what is the wingspan of the actual aircraft in metres?
   (b) If the real aircraft is 40 metres long, what is the length of the model in centimetres?

15. An architect's model of a block of flats is in the scale 1 : 50.
   (a) If the model is 0.8 metres wide, how wide is the actual block of flats?
   (b) If the block of flats is 30 metres high, how tall is the model?

16. A map of Singapore is drawn to a scale of 1 : 180 000.
   (a) In the map, the distance from Sentosa to Changi Ferry Terminal is 13.5 cm. Find the actual distance, in km, from Sentosa to Changi Ferry Termainal.
   (b) The Sentosa island has an area of 5 square kilometres. Find, in square centimetres, the area representing the island on the map.

17. The scale of a map is given as 4 cm : 1 km.
   (a) Rewrite the ratio as simply as possible.
   (b) What is the length of a river, which is measured as 3 cm on the map?
   (c) The distance between two towns is 8 km. How far apart are the towns on the map?

18. Given that 2 cm on a map represents 3 km on the ground,
   (a) calculate the distance, in km, between two towns which are 7 cm apart on the map;
   (b) express the scale of the map in the form $1 : n$;
   (c) calculate, in cm^2, the area of the map which represents a lake of area 81 km^2 on the ground.

19. A map of a region is drawn to a scale of 1 : 25 000.
   (a) Calculate the actual distance, in kilometres, represented by 24 cm on the map.
   (b) Two HDB area offices are 3.5 km apart. Calculate, in centimetres, their distance apart on the map.
   (c) On the map, a reservoir has an area of 16 cm^2. Calculate, in square kilometres, the actual area of the reservoir.

20. Given that 5 cm on a map represents 6 km on the ground.
   (a) Calculate the actual distance, in km, between two towns which are 9.5 cm apart on the map.
   (b) A road has a length of 8.4 km. Calculate its length on the map.
   (c) The area of a new township on the map is 15 cm^2. Calculate its actual area in hectares. (1 ha = 10 000 m^2)

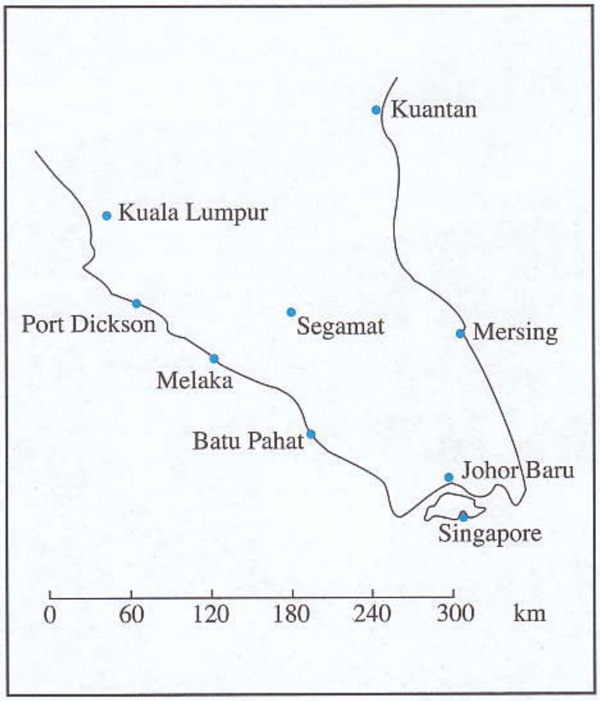
21. The plan of a shopping complex is drawn to a scale of 1 : 400.

(a) Find the length, in metres, of a corridor which is represented by a line 24.5 cm long on the plan.

(b) The area of the floor of a fast food restaurant is 400 m^2. Find its area on the plan.

(c) A supermarket on the plan occupies an area of 0.25 m^2. Calculate its actual area in hectares. (1 ha = 10 000 m^2)

22. Use the map of South Malaysia to answer the following questions. You may use a ruler to measure the approximate distances between any two places and then calculate the actual distances from the given scale.



(a) What is the distance between Singapore and Kuantan?

(b) How much would it cost to hire a taxi to go from Melaka to Kuala Lumpur if the taxi fare is 60 cents per km?

(c) A car travels at an average speed of 60 km/h from Batu Pahat to Port Dickson. How long will the journey take?

(d) A train takes 4 hours to travel from Johor Baru to Segamat. Find its average speed in km/h.

(e) An express bus travels from Kuala Lumpur to Kuantan and it charges 8 cents per km per person. How much would it cost a man to travel from Kuala Lumpur to Kuantan?

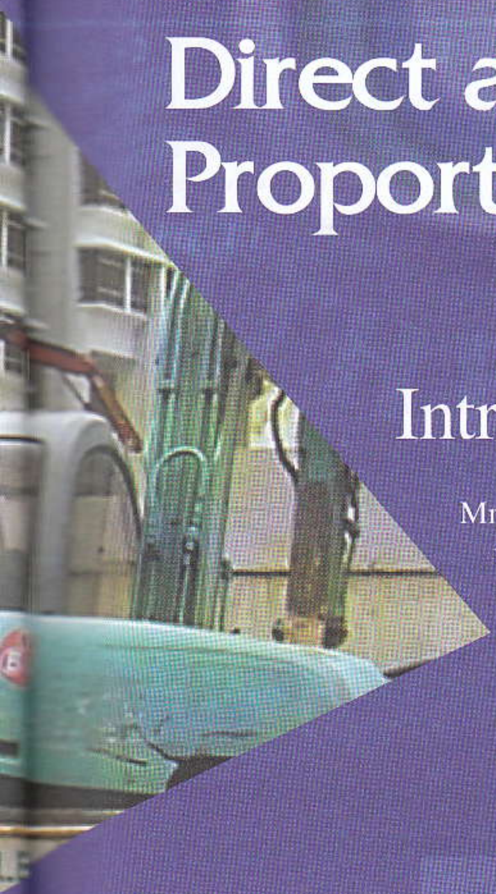
In this chapter, you will learn how to

- *write an equation connecting two quantities which are directly proportional to each other;*
- *write an equation connecting two quantities which are inversely proportional to each other;*
- *solve problems involving direct and inverse proportions.*

CHAPTER

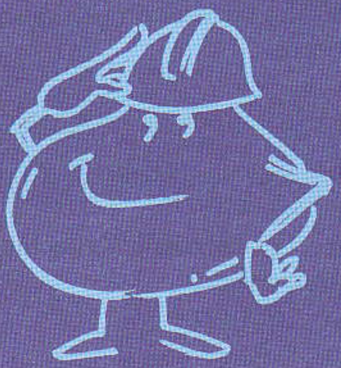
2

Direct and Inverse Proportions



Introduction

Mr Koh is the foreman of a construction project. He has been told that the project has to be completed in a shorter period of time. Given the original number of workers and period of time, how many more workers does he need?



Direct Proportion

If we borrow books from the National Library and are late in returning the books, we will be fined 15 cents per day for each overdue book. The table below shows the fines for an overdue book.

No. of days (x)	1	2	3	4	5	6	7	8	9	10
Fine (y cents)	15	30	45	60	75	90	105	120	135	150

Explorations

Based on the table shown above, answer the following questions.

(a) If the book is overdue for a greater number of days, will the fine be greater or smaller?
(b) If the number of days the book is overdue is doubled, will the fine also be doubled?
For example, compare the fines for 3 days and for 6 days.
(c) If the number of days the book is overdue is tripled, what will happen to the fine?
(d) If the number of days the book is overdue is halved, will the fine also be halved?
For example, compare the fines for 10 days and for 5 days.
(e) If the number of days the book is overdue is reduced by $\frac{1}{3}$, will the fine also be reduced by $\frac{1}{3}$?

From the above Exploration, we notice that if the number of days (x) the book is overdue increases, the fine (y cents) will also increase proportionally, i.e. if x is doubled, y will also be doubled; if x is tripled, y will also be tripled.

If the number of days (x) the book is overdue decreases, the fine (y cents) will also decrease proportionally, i.e. if x is halved, y will also be halved; if x decreases by $\frac{1}{3}$, y will also decrease by $\frac{1}{3}$.

This is called **direct proportion**. We say that the fine (y cents) is directly proportional to the number of days (x) a book is overdue.

The table below shows an additional row for the rate $\frac{y}{x}$. Complete the table below.

No. of days (x)	1	2	3	4	5	6	7	8	9	10
Fine (y cents)	15	30	45	60	75	90	105	120	135	150
Rate	$\frac{15}{1}=2$	$\frac{45}{3}=2$	$\frac{30}{2}=2$							

What do you notice about the rate $\frac{y}{x}$?

In direct proportion, the rate $\frac{y}{x}$ is a **constant.** In this case, $\frac{y}{x}=15$.

Let the number of days the book is overdue be $x_1 = 3$. Then the corresponding fine is $y_1 = 45$ (cents).

Let the number of days the book is overdue be $x_2 = 6$. Then the corresponding fine is $y_2 = 90$ (cents).

From the above table, $\frac{y_1}{x_1}=\frac{45}{5}=15$ and $\frac{y_2}{x_2}=\frac{90}{6}=15$.

$\therefore \quad \frac{y_1}{x_1}=\frac{y_2}{x_2}=15$ (constant), i.e. the two **rates** $\frac{y_1}{x_1}$ and $\frac{y_2}{x_2}$ are equal, or

rearranging, $\frac{x_2}{x_1}=\frac{y_2}{y_1}$, i.e. the two ratios $\frac{x_2}{x_1}$ and $\frac{y_2}{y_1}$ are equal.

Therefore,

If y is directly proportional to x, then $\frac{y_1}{x_1}=\frac{y_2}{x_2}$ or $\frac{x_2}{x_1}=\frac{y_2}{y_1}$.

Notes

*In Book 1, we have learnt that a **rate** is a comparison of quantities of **different** kinds. In this case, compare the fine (y cents) with the number of days overdue (x). We have also learnt that a **ratio** is a comparison of quantities of the **same** kind. In this case,* $\frac{x_2}{x_1}$ *(or* $x_2 : x_1$*) compares the number of days overdue, and* $\frac{y_2}{y_1}$ *(or* $y_2 : y_1$*) compares the fines.*

Example 1

Find the cost of 13 kg of biscuits if 6 kg of them cost \$27.

Solution

Method 1: Using Proportion

First, note that the cost of the biscuits is **directly proportional** to the mass of the biscuits.

Let \$$x$ be the cost of 13 kg of biscuits.

Then $\frac{x}{13} = \frac{27}{6}$ (recall direct proportion statement: $\frac{x_1}{y_1} = \frac{x_2}{y_2}$)

i.e. $x = \frac{27}{6} \times 13 = \$58.50$.

Alternatively, $\frac{x}{27} = \frac{13}{6}$ (recall direct proportion statement: $\frac{x_1}{x_2} = \frac{y_1}{y_2}$)

i.e. $x = \frac{13}{6} \times 13 = \$58.50$.

$\therefore$ 13 kg of biscuits cost \$58.50.

Method 2: Unitary Method

6 kg of biscuits cost \$27.

1 kg of biscuits costs $\frac{\$27}{6}$

13 kg of biscuits cost $\frac{\$27}{6} \times 13 = \$58.50$.

or simply,

6 kg $\longrightarrow$ \$27

1 kg $\longrightarrow$ $\frac{\$27}{6}$

13 kg $\longrightarrow$ $\frac{\$27}{6} \times 13 = \$58.50$

$\therefore$ 13 kg of biscuits cost \$58.50.

Notes

*This is called the **unitary** method because it involves finding the cost of 1 kg (or 1 **unit**) of biscuits first.*

Example 2

Find x if 30 : 50 = 6 : x.

Solution

$30 : 50 = 6 : x$

i.e. $\frac{30}{50} = \frac{6}{x}$

$\therefore \quad 30\,x = 6 \times 50$

$x = \frac{6 \times 50}{30} = 10$

Example 3

The amount of petrol in the fuel tanks of 3 cars A, B and C are in the ratio 8:10:13. If the tank with the smallest amount of petrol has 52 litres of petrol, find the difference in the amount of petrol in the other two cars.

Solution

Method 1: Unitary Method

The difference in the amount of petrol in car B and car C is $13 - 10 = 3$ parts.

Car A has 8 parts of petrol and this is equal to 52 litres.

$\therefore \quad$ 8 parts $\longrightarrow$ 52 litres

1 part $\longrightarrow \frac{52}{8}$ litres

3 parts $\longrightarrow \frac{52}{8} \times 3 = 19.5$ litres

$\therefore \quad$ the difference in the amount of petrol in car B and car C is 19.5 litres.

Method 2: Using Proportion

Let the amount of petrol in cars A, B and C be x, y and z respectively. Then $x : y : z = 8 : 10 : 13$

$\therefore\ x : y = 8 : 10$ and $x : z = 8 : 13$

i.e. $\frac{x}{y} = \frac{8}{10}$ and $\frac{x}{z} = \frac{8}{13}$

$\frac{52}{y} = \frac{8}{10}$ and $\frac{52}{z} = \frac{8}{13}$

$\therefore\ 8y = 52 \times 10$ and $8z = 52 \times 13$

$y = \frac{52 \times 10}{8} = 65$ and $z = \frac{52 \times 13}{8} = 84.5$

the difference in the amount of petrol in car B and car C is $84.5 - 65 = 19.5$ litres.

Exercise 2a

1. A pile of 108 identical books has a mass of 30 kg. Find
   (a) the mass of 150 books, and
   (b) the number of books that has a mass of 20 kg.

2. In a bookstore, 60 books of the same kind occupy 1.5 m of shelf length. How much shelf length is required for 300 such books? If a shelf is 80 cm long, how many such books are needed to fill the shelf completely?

3. Find the cost of
   (a) 8 books when 6 books cost $48, given that the price of each book is the same;
   (b) 10 kg of tea when 3 kg of tea cost $18;
   (c) a kg of sugar when b kg of sugar cost c dollars.

4. $\frac{5}{9}$ of a piece of metal has a mass of 7 kg. What is the mass of $\frac{2}{7}$ of the metal?

5. Find x in each of the following cases:
   (a) $4 : 7 = x : 5$
   (b) $18 : 7 = 10 : x$
   (c) $9 : x = 24 : 88$
   (d) $x : 8 = 99 : 44$
   (e) $x : 12 = 42 : 63$
   (f) $x : 32 = 250 : x$

6. Find the ratio of $x : y$ in each of the following cases:
   (a) $5x = 7y$
   (b) $3.2x = 1.2y$
   (c) $2\frac{1}{2}x = 4\frac{1}{2}y$
   (d) $1.2x = 2\frac{3}{4}y$

7. A shopowner blends three types of tea, A, B and C, in the ratio 6 : 4 : 7. If the mass of tea C in the mixture is 28 kg, find the difference in the masses of the other two types of tea.

8. The lengths of three pieces of wire are in the ratio 10 : 15 : 8. If the length of the shortest piece of wire is 2.4 m, find the difference in the lengths of the other two pieces of wire.

More on Direct Proportion

Consider the example of overdue books on page 38. We have seen that $\frac{y}{x} = 15$ which is a constant. If we represent this constant by k, then $\frac{y}{x} = k$ or $y = kx$, where $k \neq 0$.

If y is directly proportional to x, then $\frac{y}{x} = k$ or $y = kx$, where k is a constant and $k \neq 0$.

Thinking Time

If y is directly proportional to x, is x directly proportional to y? Why or why not?

Example 4

If y is directly proportional to x and if $y = 12$ when $x = 4$, find

(a) *an equation connecting x and y,*
(b) *the value of y when $x = 5$,*
(c) *the value of x when $y = 21$.*

Notes

You can also start with $k = \frac{x}{y} = \frac{12}{4} = 3$ since y is directly proportional to x.

Solution

(a) Since y is directly proportional to x, then $y = kx$, where k is a constant.
When $x = 4$, $y = 12$. So $12 = k \times 4$

i.e. $k = \frac{12}{4} = 3$.

$\therefore$ the equation is $y = 3x$.

(b) When $x = 5$, $y = 3x = 3 \times 5 = 15$.

(c) When $y = 21$, $3x = 21$

i.e. $x = \frac{21}{3} = 7$

Example 5

The expenses $E, of a tea party are directly proportional to the number of guests, N, present. For 30 guests, the expenses are $210. Find the expenses of 80 guests.

> **Notes**
>
> *Example 5 can be solved using the Unitary Method or the 'Proportion Method' (see Example 1 solution). However, there are other forms of direct proportion where the 'Unitary Method' and the 'Proportion Method' will not work. That is why we need to learn the above algebraic or functional method to solve these problems later (see Example 8).*

Solution

Since E is directly proportional to N, then $E = kN$, where k is a constant. When $N = 30$, $E = 210$. Then $210 = k \times 30$

i.e. $k = \frac{210}{30} = 7$

$\therefore \quad E = 7N.$

When $N = 80$, $E = 7N = 7 \times 80 = 560$.

$\therefore$ the expenses of 80 guests are $560.

Graphical Representation of Direct Proportion

Consider the example of overdue books on page 38 again. The table below shows the fines (y cents) for an overdue book for various number of days overdue (x) where $\frac{y}{x} = 15$ or $y = 15x$.

No. of days (x)	0	1	2	3	4	5	6	7	8	9	10
Fine in cents ($y = 15x$)	0	15	30	45	60	75	90	105	120	135	15

If we plot y against x on a sheet of graph paper (see Fig. 2.1), we will get a straight line that passes through the origin (0, 0).

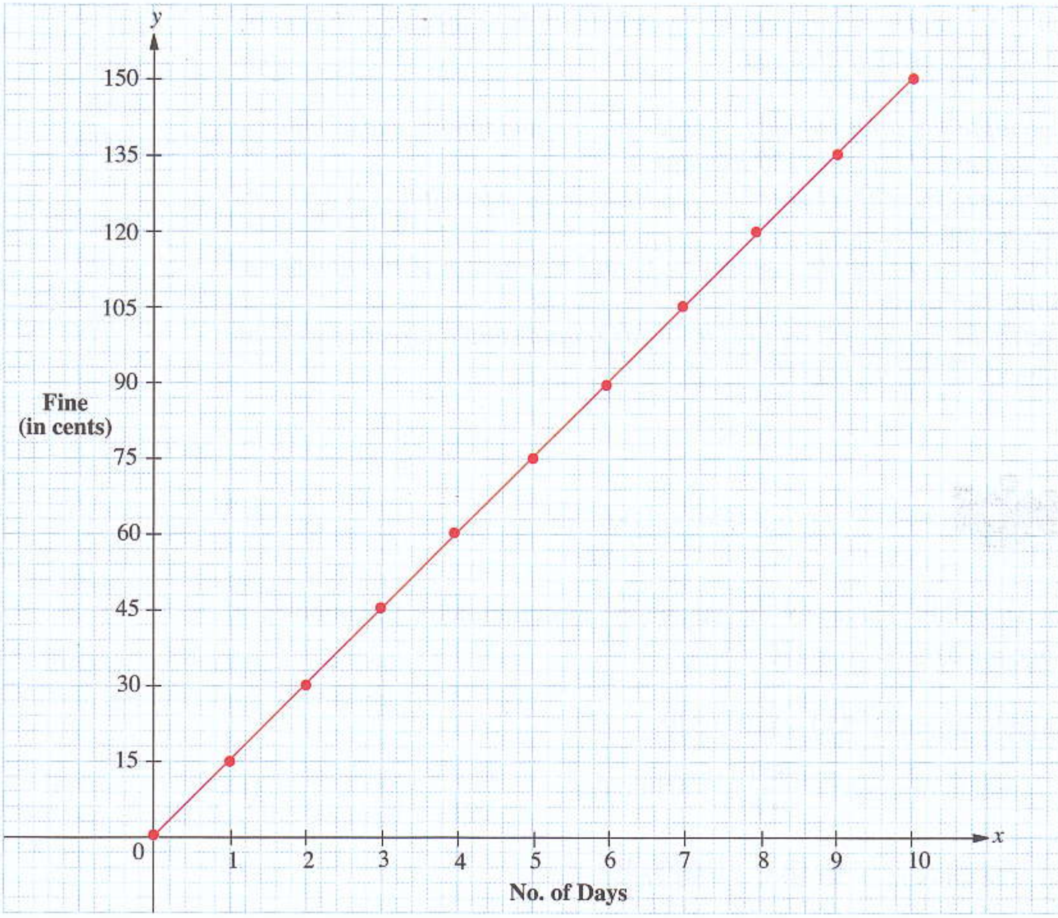


Fig. 2.1

In general

> If y is directly proportional to x, then the graph of y against x is a straight line that passes through the origin.

If we plot x against y, will we still get a straight line that passes through the origin? Why or why not?

Example 6

The total monthly charges $C for a handphone subscription consist of a fixed amount of $20 and a variable amount which depends on the usage. For every minute used, 20 cents is charged.

(a) Calculate the total monthly charges for the handphone subscription if the amount of usage is 120 minutes.
(b) If the total monthly charges for the handphone subscription is $50, find the amount of usage.
(c) Write down a formula connecting C and n, where n is the number of minutes of usage.
(d) Sketch the graph of C against n. Is this relationship a direct proportion between C and n? Explain.

Solution

(a) Total monthly charges for the handphone subscription for 120 minutes of usage = \$20 + \$0.20 × 120 = \$44.

(b) Variable amount = \$50 – \$20 = \$30.

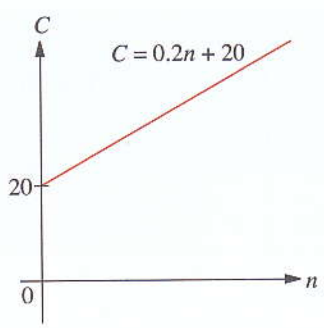
$\therefore$ amount of usage = $\dfrac{\$30}{\$0.20} = 150$ minutes

(c) Variable amount = $\$0.20 \times n = \$0.2n$

$\therefore$ total monthly charges, $\$C = \$0.2n + \$20$

i.e. $C = 0.2n + 20$

(d)



This is **not** a direct proportion between C and n because the line does not pass through the origin.

Exercise 2b

1. If y is directly proportional to x and if $y = 6$ when $x = 2$, find
   (a) an equation connecting x and y,
   (b) the value of y when $x = 10$,
   (c) the value of x when $y = 9$.

2. Given that x is directly proportional to y, and $x = 4.5$ when $y = 3$, find
   (a) a relation between x and y,
   (b) the value of x when $y = 7$,
   (c) the value of y when $x = 12$.

3. If Q is directly proportional to P and if $Q = 28$ when $P = 4$, find
   (a) a law connecting P and Q,
   (b) the value of Q when $P = 5$,
   (c) the value of P when $Q = 42$.

4. If y is directly proportional to x and if $y = 5$ when $x = 2$, find the value of y when $x = 7$.

5. Given that z is directly proportional to x, and $x = 3$ when $z = 12$, find the value of x when $z = 18$.

6. If B is directly proportional to A and if $B = 3$ when $A = 18$, find the value of B when $A = 24$.

7. Given that y is directly proportional to x, copy and complete the tables below:

   (a)

x	4	5	7		
y		15		24	34.5

   (b)

x	4	20	24		
y			6	9	11

   (c)

x	2	3	5.5		
y		3.6		9.6	11.4

8. The cost, \$$C$, of transporting goods is directly proportional to the distance, d km. Given that $C = 100$ when $d = 60$, find
   (a) an equation connecting C and d,
   (b) the cost of transporting goods for 45 km,
   (c) the distance if the cost of transporting goods is \$120.

9. The horizontal force, F Newtons, needed to push a block of metal along a horizontal surface is directly proportional to the mass, m kg, of the block. When $m = 5$, $F = 49$. Find
   (a) the force needed to push a block of metal with a mass of 14 kg,
   (b) the mass of the block of metal if the force needed to push it is 215.6 Newtons.

10. The pressure, P units, of a container of gas is directly proportional to its temperature T°C. Given that $P = 25$ when $T = 10$, find
    (a) the value of P when $T = 24$,
    (b) the value of T when $P = 12$.

11. The voltage, V volts, needed to send a fixed amount of current through a wire, is directly proportional to its resistance, R ohms. If $V = 9$ when $R = 6$, find
    (a) the value of V when $R = 15$,
    (b) the value of R when $V = 15$.

12. If y is directly proportional to x and if $y = 20$ when $x = 5$,

(a) find an equation connecting x and y,

(b) sketch the graph of y against x.

13. If z is directly proportional to y and if $z = 48$ when $y = 6$,

(a) find a law connecting y and z,

(b) sketch the graph of z against y.

14. The total monthly cost, \$$C$, of running a kindergarten with an enrolment of n children consists of a fixed amount of \$5000 and a variable amount which depends on the enrolment. For every child enrolled, the variable amount is \$41.

(a) Calculate the total monthly cost of running the kindergarten if the enrolment is 200.

(b) If the total monthly cost of running the kindergarten is \$20 580, calculate the number of children attending the kindergarten.

(c) Write down a formula connecting C and n.

(d) Sketch the graph of C against n. Is this relationship a direct proportion between C and n? Explain.

15. A company pays a salesman \$$D$ per month to sell tyres. The amount is made up of a basic salary of \$600 plus \$8 for each of the n tyres he sells each month.

(a) Calculate the salesman's income for the month when he sold 95 tyres.

(b) If in a particular month he received \$1680, find the number of tyres he sold.

(c) Write down a formula connecting D and n.

(d) Sketch the graph of D against n. Is this relationship a direct proportion between D and n? Explain.

16. An ice manufacturing machine requires 10 minutes to warm up before the production of ice begins. The mass, in tonnes, of ice produced is directly proportional to the number of hours of production. Given that 20 tonnes of ice are produced when the machine ran for half an hour, find the mass of ice manufactured when the machine ran for $1\frac{3}{4}$ hours.

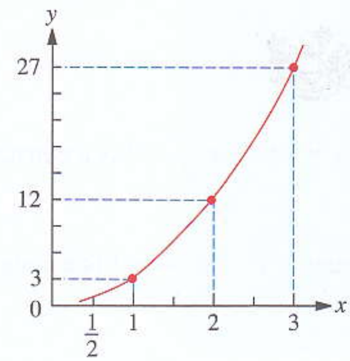
Other Forms of Direct Proportion

Explorations

The variables x and y are related by the equation $y = 3x^2$.

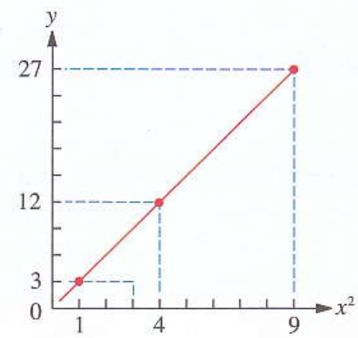
(a) Is y directly proportional to x? Why or why not?

(b) The table below shows some values of y against x. Does it show a direct proportion between x and y? Why or why not?



x	$\frac{1}{2}$	1	2	3
y	$\frac{3}{4}$	3	12	27
$\frac{y}{x}$	$\frac{3}{2}$	3	6	9

(c) Now let us try to plot y against x^2.
The table below shows some values of y against x^2.



x	$\frac{1}{2}$	1	2	3
x^2	$\frac{1}{4}$	1	4	9
y	$\frac{3}{4}$	3	12	27
$\frac{y}{x^2}$	3	3	3	3

Does it show a direct proportion between x^2 and y?

From the above Exploration, when $y = 3x^2$, y is not directly proportional to x because $\frac{y}{x} = 3x$ is not a constant. But y is directly proportional to x^2 because $\frac{y}{x^2} = 3$ is a constant. If you are confused, you can always let $X = x^2$. Then $y = 3x^2$ becomes $y = 3X$, i.e. $\frac{y}{X} = 3$. So y is directly proportional to X ($= x^2$).

Example 7

For each of the following equations, state which two variables are directly proportional to each other and explain why.

(a) $y = 5x^3$,

(b) $y = \sqrt{x}$,

(c) $A = \pi r^2$ *where A is the area of a circle of radius r and* π *is a constant,*

(d) $y^2 = 3x$,

(e) $y - 1 = 4x$

Solution

(a) Since $y = 5x^3$ or $\frac{y}{x^3} = 5$ is a constant, then y is directly proportional to x^3.

(b) Since $y = \sqrt{x}$ or $\frac{y}{\sqrt{x}} = 1$ is a constant, then y is directly proportional to $\sqrt{x}$.

(c) Since $A = \pi r^2$ or $\frac{A}{r^2} = \pi$ is a constant, then A is directly proportional to r^2.

(d) Since $y^2 = 3x$ or $\frac{y^2}{x} = 3$ is a constant, then y^2 is directly proportional to x.

(e) Since $y - 1 = 4x$ or $\frac{y-1}{x} = 4$ is a constant, then $y - 1$ is directly proportional to x.

Do it Yourself

In Example 6, we know that the formula connecting C and n is $C = 0.2n + 20$ and it does not show a direct proportion between n and C. But n is actually directly proportional to some variable. What is the variable?

Example 8

y is directly proportional to x^2 and $y = 20$ when $x = 2$,

(a) find an equation connecting x and y,
(b) calculate the value of y when $x = 3$,
(c) calculate the values of x when $y = 1.25$,
(d) sketch the graph of y against x^2.

Solution

(a) Since y is directly proportional to x^2, then $y = kx^2$, where k is a constant.

When $x = 2, y = 20$. So $20 = k(2^2)$

i.e. $4k = 20$

$$k = \frac{20}{4} = 5$$

$\therefore$ the equation is $y = 5x^2$.

(b) When $x = 3$, $y = 5x^2 = 5(3^2) = 45$.

(c) When $y = 1.25$, $5x^2 = 1.25$

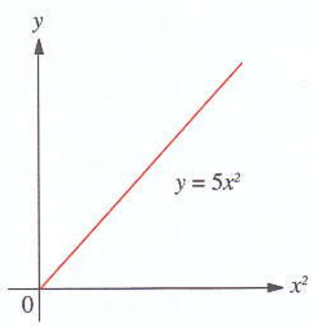
i.e. $x^2 = \frac{1.25}{5} = 0.25$

$\therefore \quad x^2 = \pm\sqrt{0.25}$

$= \pm 0.5$

(d) Since y is directly proportional to x^2, then the graph of y against x^2 is a straight line that passes through the origin. But x^2 cannot be negative. So the graph must start from the origin.

The graph is shown below.



Notes

It is difficult to use the Unitary Method or the 'Proportion Method' (see Example 1 solution) to solve this type of direct proportion. That is why you need to learn and use the algebraic or functional method (see Example 5 note against this).

Example 9

The volume of a solid, V, is directly proportional to the cube of its radius, r. Given that its volume is 905 cm³ when its radius is 6 cm, find

(a) an equation relating V and r,
(b) its volume when its radius is 10 cm.

Solution

(a) Let $V = kr^3$ where k is a constant.

When $V = 905$, $r = 6$

$$905 = k(6)^3$$

$$k = \frac{905}{6^3}$$

$$= 4.19 \text{ (correct to 3 sig. fig.)}$$

$\therefore$ the equation is $V = 4.19\, r^3$

(b) When $r = 10$, $V = 4.19 \times 10^3$

$$= 4190 \text{ cm}^3$$

$\therefore$ the volume of the solid is 4190 cm³.

Exercise 2c

1. For each of the following equations, state which two variables are directly proportional to each other and explain why.

(a) $y = 4x^2$

(b) $y = 3\sqrt{x}$

(c) $y^2 = 5x^3$

(d) $q = 7p^2$

(e) $q = 2p\sqrt{p}$

(f) $m^3 = n^2$

(g) $n = 3(m-1)^2$

(h) $y - 2 = 4(x+1)^3$

(i) $y = 3x + 1$

(j) $2y = x - 3$

(k) $y = \frac{1}{x}$

(l) $y = \frac{2}{x}$

2. If y is directly proportional to x^2 and if $y = 18$ when $x = 3$,
   (a) find an equation connecting x and y,
   (b) calculate the value of y when $x = 5$,
   (c) calculate the values of x when $y = 32$,
   (d) sketch the graph of y against x^2.

3. Given that x is directly proportional to y^3, and $x = 32$ when $y = 2$,
   (a) find a relationship between x and y,
   (b) calculate the value of x when $y = 6$,
   (c) calculate the value of y when $x = 108$,
   (d) sketch the graph of x against y^3.

4. Given that z^2 is directly proportional to w, and $z = 4$ when $w = 8$,
   (a) find a law connecting w and z,
   (b) calculate the values of z when $w = 18$,
   (c) calculate the value of w when $z = 5$,
   (d) sketch the graph of z^2 against w.

5. If Q is directly proportional to $\sqrt{P}$ and if $Q = 21$ when $P = 9$, find the value of Q when $P = 81$.

6. Given that y^2 is directly proportional to the cube of x and y is always positive, find the value of y when $x = 9$ if $y = 8$ when $x = 4$.

7. Given that q is directly proportional to $(P - 1)^2$, and P is always positive, find the value of P when $q = 80$, if $q = 20$ when $P = 3$.

8. Given that y is directly proportional to the square of x, copy and complete the table below:

x	2	3	5		
y		81		441	56.25

9. Given that v is directly proportional to r^3, copy and complete the table below:

r	3	4	6		
v			648	375	1029

10. Given that the mass, m g, of a sphere is directly proportional to the cube of its radius, r cm, copy and complete the table below:

r	0.2	0.7		1.5	
m			0.25	6.75	11.664

11. The length, l cm, of a simple pendulum is directly proportional to the square of its period (time to swing to and fro), T seconds. A pendulum with a length of 220.5 cm has a period of 3 s.
    (a) Find an equation connecting l and T.
    (b) Calculate the length of a pendulum whose period is 5 s.
    (c) What is the period of a pendulum whose length is 0.98 m?

12. Within a certain period of its life, the length, L, of an earthworm is directly proportional to the square root of N, the number of hours after its birth. If an earthworm is 2.5 cm long after 1 hour, how long will it be after 4 hours? How long will it take to grow to a length of 15 cm?

Inverse Proportion

The time taken by a car to travel a distance of 120 km at various speeds is shown in the table below:

Speed (x km/h)	10	20	30	40	60	120
Time taken (y hours)	12	6	4	3	2	1

Explorations

(a) If the speed of the car becomes greater, will the time taken to travel the distance of 120 km be greater or smaller?

(b) If the speed of the car is doubled, will the time taken be halved? For example, compare $x = 20$ and $x = 40$.

(c) If the speed of the car is tripled, will the time taken be reduced by a factor of $\frac{1}{3}$?

(d) If the speed of the car is halved, what will happen to the time taken? For example, compare $x = 60$ and $x = 30$.

(e) If the speed of the car is reduced by $\frac{1}{3}$, what will happen to the time taken?

We will notice that if the speed of the car, x km/h, **increases**, the time taken, y hours, to travel the distance of 120 km will **decrease proportionally**, i.e. if x is doubled, y will be halved; if x is tripled, y will be reduced to $\frac{1}{3}$ its original value. If the speed **decreases**, the time taken will **increase proportionally**, i.e. if x is halved, y will be doubled; if x is reduced by $\frac{1}{3}$, y will be tripled. This is called **inverse proportion**. We say that the speed (x km/h) is inversely proportional to the time taken (y hours).

The table below shows an additional row for the product xy. Complete the table below.

Speed (x km/h)	10	20	30	40	60	120
Time taken (y hours)	12	6	4	3	2	1
Product xy	10×12 $= 120$	20×6 $= 120$				

What do you notice about the product xy?

In inverse proportion, the product xy is a constant. In this case, $xy = 120$ = distance travelled.

Let the speed be $x_1 = 20$. Then the corresponding time taken is $y_1 = 6$.

Let the speed be $x_2 = 40$. Then the corresponding time taken is $y_2 = 3$.

From the above table, $x_1y_1 = 20 \times 6 = 120$ and $x_2y_2 = 40 \times 3 = 120$.

$\therefore \quad x_1y_1 = x_2y_2 = 120$ (constant)

or $\dfrac{y_2}{y_1} = \dfrac{x_1}{x_2}$

You have learnt that $\frac{y_2}{y_1} = \frac{x_2}{x_1}$ for direct proportion.

Hence, for inverse proportion, $\frac{y_2}{y_1} = \frac{x_1}{x_2}$ or $x_1y_1 = x_2y_2$. Note the order of x_1 and x_2.

In summary

> If y is inversely proportional to x, then $\dfrac{y_2}{y_1} = \dfrac{x_1}{x_2}$ or $x_1y_1 = x_2y_2$

Example 10

Ten identical pipes can fill a tank in 4 hours. How long will it take 8 of these pipes to fill the same tank?

Solution

Method 1: Use Proportion

First, note that the time taken to fill the tank is inversely proportional to the number of pipes used because the more pipes there are, the faster it takes to fill the tank.

Let x be the number of hours 8 pipes will take to fill the tank. Then
$8x = 10 \times 4$ (recall inverse proportion statement: $x_1y_1 = x_2y_2$)

ie. $x = \dfrac{10 \times 4}{8} = 5$

$\therefore$ 8 pipes can fill the tank in 5 h.

Method 2: Unitary Method

10 pipes can fill a tank in 4 hours.
1 pipe can fill the same tank in $4 \text{ h} \times 10$ (fewer pipes, require more time, thus you multiply)

8 pipes can fill the same tank in $\dfrac{4 \text{ h} \times 10}{8}$ (more pipes, require less time, thus you divide)

$= 5 \text{ h}$

or simply,

10 pipes $\longrightarrow$ 4 h
1 pipe $\longrightarrow$ $4 \text{ h} \times 10$
8 pipes $\longrightarrow$ $\dfrac{4 \text{ h} \times 10}{8} = 5 \text{ h}$

$\therefore$ 8 pipes can fill the tank in 5 h.

> **Problem Solving Tips**
>
> *It is more confusing to use the inverse proportion statement, $\frac{y_2}{y_1} = \frac{x_1}{x_2}$, to solve this type of problems because you must switch the order of x_1 and x_2.*

Example 11

It takes 5 men to paint 2 identical houses in 3 days. Assuming that all the men work at the same rate, how long will it take 10 men to paint 8 of these houses.

Solution

There are 3 variables in this question: number of men, number of houses and number of days. The trick is to keep one of these variables constant at a time.

First, we keep the number of houses constant.

Number of men	Number of houses	Number of days	
5	2	3	
1	2	3×5	(fewer men require more days, thus you multiply)
10	2	$\frac{3 \times 5}{10} = 1.5$	(more men require fewer days, thus you divide)

Notes

You multiply for direct proportion and divide for inverse proportion.

Next, we keep the number of men constant.

Number of men	Number of houses	Number of days	
10	2	1.5	
10	1	$\frac{1.5}{2}$	(fewer houses require fewer days, thus you divide)
10	8	$\frac{1.5}{2} \times 8 = 6$	(more houses require more days, thus you multiply)

$\therefore$ 10 men will take 6 days to paint 8 houses.

Exercise 2d

1. Which of the following are in inverse proportion?
   (a) The number of pencils you buy and their total cost.
   (b) The number of pipes filling a tank and the time taken to fill it.
   (c) The number of men doing a job and the time taken to finish it.
   (d) The number of cattle to be fed and the amount of feed, assuming each cattle eats the same amount.
   (e) The number of cattle to be fed and the time taken to finish a certain amount of the feed.

2. Four pipes can fill a tank in 70 minutes. How long will it take to fill the tank if 7 pipes are used?

3. A school librarian has enough money to order 8 paperback books at $5.50 each. If the librarian decides instead to order books with hard covers at $8.80 each, how many such books can the librarian buy?

4. Thirty-five workers are needed to build a house in 16 days. How many days will 28 workers working at the same rate take to build the same house?

5. An aircraft flying at an average speed of 770 km/h takes 15 hours to complete a journey. Find the time taken for the aircraft to complete the same journey if its average speed is 660 km/h.

6. A consignment of fodder lasts 1260 cattle for 50 days. Given that the cattle consume the fodder at a constant rate, find
   (a) the number of cattle an equal consignment of fodder lasts for 75 days;
   (b) the number of days an equal consignment of fodder lasts 1575 cattle.

7. A contractor estimates that he would need 56 workers to complete a job in 21 days. If he is asked to complete the job in 14 days, find the additional number of workers he has to employ. Assume that the workers all work at the same rate.

8. At a scouts' camp, there is sufficient food to last 72 scouts for 6 days. If 18 scouts do not turn up for the camp, how much longer can the food last for the other scouts?

9. It takes 3 men to dig 2 identical trenches in 5 hours. If all the men work at the same rate, how long will it take 5 men to dig 7 of these trenches?

10. It takes 12 men to make 12 tables in 9 hours. How long will it take 8 men to make 32 tables?

11. It takes 7 identical pipes to fill 3 identical tanks in 45 minutes. How long will it take 5 of the pipes to fill one of these tanks?

12. A consignment of fodder can last 1000 sheep for 20 days. If the sheep consume the fodder at the same rate, how many consignments of fodder are needed to last 550 sheep for 400 days?

More on Inverse Proportion

Let us consider again the example of the time taken by a car to travel a distance of 120 km at various speeds discussed on page 53. We saw that $xy = 120$ which is a constant. If we represent this constant by k, then $xy = k$ or $y = \frac{k}{x}$, where $k \neq 0$.

If y is inversely proportional to x, then $xy = k$ or $y = \frac{k}{x}$, where k is a constant and $k \neq 0$.

Thinking Time

If y is inversely proportional to x, is x inversely proportional to y? Why or why not?

Example 12

If y is inversely proportional to x and y = 3 when x = 4, find
(a) an equation connecting x and y,
(b) the value of y when x = 6,
(c) the value of x when y = 48.

Solution

(a) Since y is inversely proportional to x, then $y = \frac{k}{x}$, where k is a constant.

When $x = 4$, $y = 3$. So $3 = \frac{k}{4}$

i.e. $k = 3 \times 4 = 12$.

$\therefore$ the equation is $y = \frac{12}{x}$

(b) When $x = 6$, $y = \frac{12}{x} = \frac{12}{6} = 2$.

(c) When $y = 48$, $48 = \frac{12}{x}$

i.e. $x = \frac{12}{48} = \frac{1}{4}$.

Just For Fun

1. A toy train measures $\frac{1}{4}$ m long. How long will it take for the train to pass a 1 m long bridge if it moves at a constant speed of $\frac{1}{4}$ m/s?

2. Another toy train measures $\frac{1}{2}$ m long. If it moves at a constant speed of $\frac{1}{2}$ m/s, how long will it take to pass a bridge that is $\frac{1}{2}$ m long?

3. A third toy train measures x m long. How long will it take for the train to pass a bridge that is x m long if it moves at a constant speed of x m/s?

Example 13

Boyle's Law states that the volume, V cm³, of a fixed mass of gas at constant temperature is inversely proportional to its pressure, P units. Given that the volume of a fixed mass of gas at constant temperature is 1000 cm³ when its pressure is 50 units, find the volume of the gas when its pressure is 1250 units.

Solution

Since V is inversely proportional to P, then $V = \frac{k}{P}$, where k is a constant.

When $P = 50$, $V = 1000$. So $1000 = \frac{k}{50}$

i.e. $k = 1000 \times 50 = 50\ 000$.

$$\therefore \quad V = \frac{50\ 000}{P}$$

When $P = 1250$, $V = \frac{50\ 000}{P} = \frac{50\ 000}{1250} = 40$.

$\therefore$ The volume of the gas at a pressure of 1250 units is 40 cm³.

Graphical Representation of Inverse Proportion

Recall the example of the car discussed on page 53. The table below shows the time, y hours, taken by the car to travel a distance of 120 km at various speeds, x km/h, where $xy = 120$ or $y = \frac{120}{x}$.

The table below shows some values of y against x.

Speed (x km/h)	10	20	30	40	50	60	70	80	90	100	110	20
Time taken (y hours)	12	6	4	3	2.4	2	1.7	1.5	1.3	1.2	1.1	1

Using the values of x and y, we can plot a graph as below.

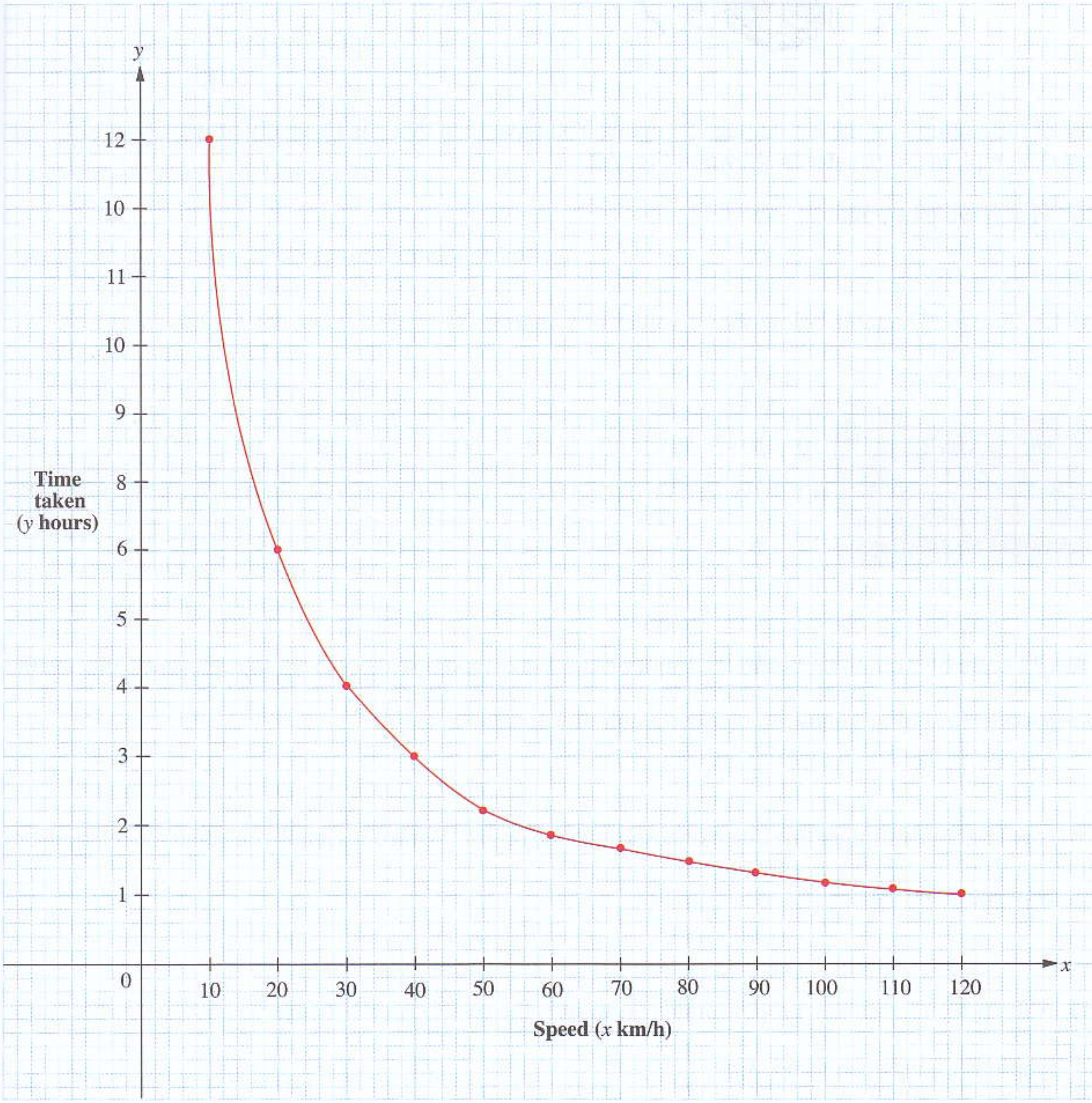


Fig. 2.2

You can see from the graph that, if x is doubled, y is halved. If x is tripled, y is reduced by $\frac{1}{3}$.

Explorations

Recall the example of the car on page 53 again. You have seen that y is inversely proportional to x, and that the graph of y against x as shown in Fig. 2.2.

But what if we plot y against $\frac{1}{x}$?

(a) Let $X = \frac{1}{x}$. The table below shows values of y against $X\left(=\frac{1}{x}\right)$

Speed (x km/h)	10	20	30	40	50	60	70	80	90	100	110	120
$X = \frac{1}{x}$	0.1	0.05	0.033	0.025	0.02	0.017	0.014	0.013	0.011	0.01	0.009	0.008
Time taken (y hours)	12	6	4	3	2.4	2	1.7	1.5	1.3	1.2	1.1	1

Using a scale of 2 cm to represent 0.01 unit on the X-axis and a scale of 2 cm to represent 1 unit on the y-axis on a sheet of graph paper, plot the graph of y against $X\left(=\frac{1}{x}\right)$.

(b) Is your graph a straight line that passes through the origin? What does it tell you about the relationship between y and $X\left(=\frac{1}{x}\right)$?

(c) Although y is inversely proportional to x, what is the relationship between y and X, i.e., what is the relationship between y and $X\left(=\frac{1}{x}\right)$?

(d) Try to write an equation connecting y and X, or an equation connecting y and $\frac{1}{x}$. What does it tell you about the relationship between y and $X\left(=\frac{1}{x}\right)$?

Exercise 2e

1. If y is inversely proportional to x, and if $y = 5$ when $x = 2$, find
   (a) an equation connecting x and y,
   (b) the value of y when $x = 8$,
   (c) the value of x when $y = 10$.

2. Given that x is inversely proportional to y, and $x = 40$ when $y = 5$, find
   (a) a relation between x and y,
   (b) the value of x when $y = 25$,
   (c) the value of y when $x = 400$.

3. If z is inversely proportional to x, and if $z = 0.25$ when $x = 2$, find
   (a) a law connecting z and x,
   (b) the value of z when $x = 5$,
   (d) the value of x when $z = 0.2$.

4. If P is inversely proportional to Q and if $P = 9$ when $Q = 2$, find the value of P when $Q = 3$.

5. Given that y is inversely proportional to x, and $y = 5$ when $x = 7$, find the value of x when $y = 70$.

6. If n is inversely proportional to m, and if $n = 3.5$ when $m = 2$, find the value of m when $n = 5$,

7. Given that y is inversely proportional to x, copy and complete the table below:

   (a)

x	0.5	2	3		
y		2		0.8	4

   (b)

x	2	2.5	3		
y			4	24	1.5

   (c)

x	3	25		4	
y			8	9	2.5

8. The frequency f of a radio wave is inversely proportional to its wavelength w. When $w = 3000$ m and $f = 100$ kHz (kHz = kilohertz), find
   (a) an equation connecting f and w,
   (b) the frequency of a radio wave if its wavelength is 500 m,
   (c) the wavelength of a radio wave if its frequency is 800 kHz.

9. The current, I amperes, passing through a wire is inversely proportional to its resistance R ohms. If $I = 12$ when $R = 0.5$, find
   (a) the current passing through the wire if its resistance is 3 ohms,
   (b) the resistance of the wire if the current passing through it is 3 amperes.

10. The pressure, P Pascals, of a fixed mass of gas at constant temperature is inversely proportional to its volume V m^3. When $V = 4$, $P = 250$. Find
   (a) the value of P when $V = 5$,
   (b) the value of V when $P = 125$.

11. The time required to complete a certain job, t hours, is inversely proportional to the number of workers.
   (a) If we need to complete the job in $\frac{3}{4}$ hour, how many workers are required?
   (b) Find the time taken to complete the job when 6 men are available.

12. If y is inversely proportional to x and if $y = 2$ when $x = 1$, find an equation connecting x and y.

13. Given that z is inversely proportional to x, and $z = 3$ when $x = 2$, find a relation between x and z.

Other Forms of Inverse Proportion

We have learnt that y is inversely proportional to x if

$$xy = k \text{ or } y = \frac{k}{x}, \text{ where } k \text{ is a constant and } k \neq 0.$$

Similarly, if y is inversely proportional to x^2, then

$$x^2y = k \text{ or } y = \frac{k}{x^2}, \text{ where } k \text{ is a constant and } k \neq 0.$$

Also, we can always let $X = x^2$. Then y is inversely proportional to X ($= x^2$) if

$$Xy = k, \text{ i.e. } x^2y = k,$$

$$\text{or } y = \frac{k}{X}, \text{ i.e. } y = \frac{k}{x^2}.$$

Example 14

For each of the following equations, state which two variables are inversely proportional to each other and explain why.

(a) $y=\dfrac{2}{x^3}$

(b) $q=\dfrac{3}{\sqrt{p}}$

(c) $n^3=\dfrac{1}{m^2}$

(d) $y=\dfrac{4}{x+2}$

(e) $y=x$

Solution

(a) Since $y=\dfrac{2}{x^3}$ or $x^3y=2$ is a constant, then y is inversely proportional to x^3.

(b) Since $q=\dfrac{3}{\sqrt{p}}$ or $q\sqrt{p}=3$ is a constant, then q is inversely proportional to $\sqrt{p}$.

(c) Since $n^3=\dfrac{1}{m^2}$ or $n^3m^2=1$ is a constant, then n^3 is inversely proportional to m^2.

(d) Since $y=\dfrac{4}{x+2}$ or $y(x+2)=4$ is a constant, then y is inversely proportional to $x+2$.

(e) Since $y=x$ or $\dfrac{y}{x}=1$, i.e., $y\left(\dfrac{1}{x}\right)=1$ is a constant, then y is inversely proportional to $\dfrac{1}{x}$.

Example 15

If y is inversely proportional to $\sqrt{x}$ *and if* $y = 6$ *when* $x = 4$,

(a) *find an equation connecting x and y,*
(b) *calculate the value of y when* $x = 16$,
(c) *calculate the value of x when* $y = 4$.

Solution

(a) Since y is inversely proportional to $\sqrt{x}$, then $y = \frac{k}{\sqrt{x}}$, where k is a constant.

When $x = 4$, $y = 6$. So $6 = \frac{k}{\sqrt{4}}$

i.e. $\frac{k}{2} = 6$

$k = 6 \times 2 = 12$

$\therefore$ the equation is $y = \frac{12}{\sqrt{x}}$.

(b) When $x = 16$, $y = \frac{12}{\sqrt{x}} = \frac{12}{\sqrt{16}} = \frac{12}{4} = 3$

(c) When $y = 4$, $4 = \frac{12}{\sqrt{x}}$

i.e. $\sqrt{x} = \frac{12}{4} = 3$

$x = 3^2 = 9$

Notes

The square root symbol refers to the positive square root. So $\sqrt{16} = 4$. *But if* $x^2 = 16$, *then* $x = \pm\sqrt{16} = \pm 4$.
i.e. if we want both the positive and the negative square roots, we must write $\pm\sqrt{}$.

Example 16

In an experiment, a drug is added to two identical flasks, each containing the same amount of a certain bacteria. The drug is then allowed to react with the bacteria for various times in t hours. It is found that the amount of bacteria left, s units, varies inversely as (t – 2) hours. In one flask, there were 6 units of bacteria left after 5 hours. Calculate the amount of units of bacteria left in another flask after 7 hours.

Solution

Since s is inversely proportional to $t-2$, then $s = \frac{k}{t-2}$, where k is a constant.

When $t = 5$, $s = 6$.

So $6 = \frac{k}{5-2}$

i.e. $\frac{k}{3} = 6$

$k = 6 \times 3 = 18$

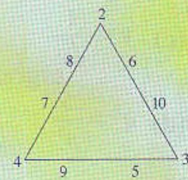
$\therefore$ the equation is $s = \frac{18}{t-2}$.

When $t = 7$, $s = \frac{18}{t-2} = \frac{18}{7-2} = \frac{18}{5} = 3.6$

$\therefore$ the amount of bacteria left in the flask after 7 hours is 3.6 units.

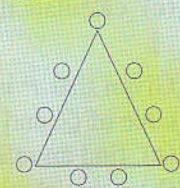
Just for Fun

1. *The sum of the numbers on each side of the triangle given below is 21.*



Rearrange the numbers such that the sum on each side is now 24.

2. *Write each of the digits 1 to 9 in each circle below so that the sum of the numbers on each side of the triangle is 20.*



3. *Write each of the digits 1 to 11 in each circle below so that the sum of the numbers along any straight line is 18.*

Exercise 2f

1. For each of the following equations, state which two variables are inversely proportional to each other and explain why.

   (a) $y=\frac{3}{x^2}$

   (b) $y=\frac{4}{x^3}$

   (c) $y=\frac{1}{\sqrt{x}}$

   (d) $q^2=\frac{5}{p^3}$

   (e) $n=\frac{2}{m\sqrt{m}}$

   (f) $z=\frac{7}{w-1}$

   (g) $y=\frac{4}{(x+1)^2}$

   (h) $y+3=\frac{1}{x^2}$

   (i) $y=\frac{4}{x^2}-1$

   (j) $2y^3=\frac{3}{x}+4$

   (k) $y=6x$

   (l) $y^2=7\sqrt{x}$

2. If y is inversely proportional to x^2, and if $y = 2$ when $x = 4$,
   (a) find an equation connecting x and y,
   (b) calculate the value of y when $x = 6$,
   (c) calculate the value of x when $y = 8$.

3. Given that P is inversely proportional to the cube of Q, and $P = 50$ when $Q = 2$,
   (a) find an relation between P and Q,
   (b) calculate the value of P when $Q = 4$,
   (c) calculate the value of Q when $P = 3.2$.

4. If z is inversely proportional to $\sqrt{x}$, and if $z = 9$ when $x = 9$,
   (a) find a law connecting x and z,
   (b) calculate the value of z when $x = 16$,
   (c) calculate the value of x when $z = 3$.

5. Given that B^2 is inversely proportional to $A + 3$ and B is always positive, find the value of B when $A = 17$ if $B = 5$ when $A = 2$.

6. If s is inversely proportional to the positive square root of t, copy and complete the table below:

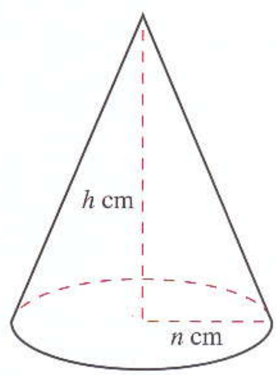
t	1	4			16
s	8		$1\frac{1}{3}$	16	

7. Given that y is inversely proportional to x^n, find the value of n and then copy and complete the table below:

x	1	2	4	
y	80	10		$\frac{1}{100}$

8. It is given that the force, F Newtons, between two particles is inversely proportional to the square of the distance, d metres, between them. If $F = 10$ when $d = 2$,
   (a) find an equation connecting F and d,
   (b) find the value of F when $d = 5$,
   (c) find the value of d when $F = 25$.

9. For a given volume, the height h cm of a cone is inversely proportional to the square of the base radius r cm. Find the height of a cone of base radius 3 cm if it has the same volume as a cone of base radius 6 cm and height 5 cm.



SUMMARY

1. A proportion is a statement expressing the equivalence of two rates or two ratios.

2. If y is directly proportional to x, then

   (a) $\frac{y}{x} = k$ or $y = kx$, where k is a constant and $k \neq 0$;

   (b) the graph of y against x is a straight line that passes through the origin.

3. If y is inversely proportional to x, then

   $xy = k$ or $y = \frac{k}{x}$, where k is a constant and $k \neq 0$.

Review Examples 2

Example 1

Given that y is directly proportional to the square of x and that $y = 16$ when $x = 2$, calculate the value of y when $x = 5$.

Solution

Since y is directly proportional to x^2, then $y = kx^2$, where k is a constant.

When $x = 2$, $y = 16$. So $16 = k(2^2)$

i.e. $4k = 16$

$$k = \frac{16}{4} = 4$$

$$\therefore \quad y = 4x^2.$$

When $x = 5$, $y = 4x^2 = 4(5^2) = 100$.

Example 2

Six men can complete a certain job in 8 hours. Suppose all the men work at the same speed, how long will 18 men take to complete the same job?

Solution

Suppose x men take y hours to finish the job.

Since y is inversely proportional to x, then $y = \frac{k}{x}$, where k is a constant.

When $x = 6$, $y = 8$. So $8 = \frac{k}{6}$

i.e. $k = 8 \times 6 = 48$.

$\therefore \quad y = \frac{48 \text{ h}}{18}$.

When $x = 18$, $y = \frac{48 \text{ h}}{18} = 2\frac{2}{3}$.

$\therefore$ it will take 18 men $2\frac{2}{3}$ hours to complete the same job.

Review Questions 2

1. If y is directly proportional to x, and if $y = 6$ when $x = 2$,
   (a) express y in terms of x,
   (b) find the value of x when $y = 12$.

2. Given that R is directly proportional to S and that $R = 140$ when $S = 5$, find the constant of proportion and calculate the value of S when $R = 170$.

3. Given that A is directly proportional to B and that $A = 1\frac{2}{3}$ when $B = \frac{5}{6}$, find A when $B = \frac{1}{3}$ and B when $A = 7\frac{1}{2}$.

4. The monthly telephone charges for a household, \$$C$, is given by the formula

$$C = a + bn,$$

where n is the number of units of time during which the telephone is used, and a and b are constants. When 300 units of time are used, the charges are \$29 and when 700 units of time are used, the charges are \$57.
   (a) Write down two equations in a and b.
   (b) Find the values of a and b by solving these equations.
   (c) Find the monthly charges if the telephone is used for 320 units of time.

5. If d is directly proportional to t^2, and if $d = 27$ when $t = 3$, find an equation giving d in terms of t.

6. If V is directly proportional to x^3 and if $V = 108$ when $x = 3$, find V in terms of x. Find also V when $x = 6$ and x when $V = 4000$.

7. Given that x is directly proportional to y^2, and that $x = 9\frac{3}{8}$ when $y = 2\frac{1}{2}$, calculate the value of x when $y = 3$.

8. If x is directly proportional to $\sqrt[3]{v}$ and if $x = 4$ when $v = 64$, find the value of x when $v = 125$ and the value of v when $x = 2$.

9. Given that y is directly proportional to the square of x and that $y = 112$ when $x = 4$, calculate
   (a) the value of y when $x = 2$,
   (b) the values of x when $y = 700$.

10. Given that the mass m of a cube is directly to the cube of its edge x and that $m = 24$ when $x = 2$, find m when $x = 3$.

11. The value of a mirror exceeding 10 m^2 in area varies directly as the square of its area. Given that a 60-m^2 mirror costs \$400, find the price of a 45-m^2 mirror.

12. If y is inversely proportional to x and if $y = 4$ when $x = 3$,
    (a) express y in terms of x,
    (b) find the value of y when $x = 6$.

13. If z is inversely proportional to $\sqrt{x}$, and if $z = 6$ when $x = 9$,
    (a) express z in terms of x,
    (b) find the value of z when $x = 25$.

14. Given that N is inversely proportional to r^2 and that $N = 3$ when $r = 5$, find the value of N when $r = 10$.

15. If y is directly proportional to x^2 and if the difference in the values of y when $x = 1$ and $x = 3$ is 32, find the value of y when $x = -2$.

16. A man donates a certain amount of money to a charity organisation each month. His monthly donation is directly proportional to the square of his monthly savings. Given that he saved \$900 and \$1200 in January and February 2006 respectively, and that his donation increased by \$35 in February, find the amounts he donated to the charity organisation in January and February 2006.

17. Given the table of values for s and t, write down a formula expressing s in terms of t for these values:

t	0	1	2	3
s	0	5	20	45

18. It is given that the force between two particles is inversely proportional to the square of the distance between them. If the force is F when the distance between them is r, and cF when the distance is $5r$, write down the value of c.

19. 5 men are hired to complete a certain job. If an additional man is hired, the job can be completed 8 days earlier. Given that the number of days required to complete the job is inversely proportional to the number of men hired, find the number of additional men who must be hired in order for the job to be completed 28 days earlier.

In this chapter, you will learn how to

- *expand products of algebraic expressions;*
- *factorise algebraic expressions;*
- *recognise and apply three algebraic identities;*
- *solve quadratic equations by factorisation;*
- *solve problems involving quadratic equations.*

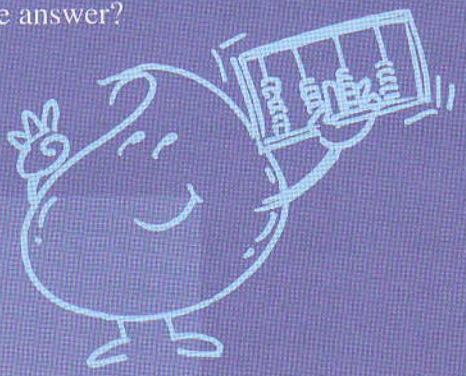
CHAPTER

3

Expansion and Factorisation of Algebraic Expressions

Introduction

The children are learning how to use the abacus to help them calculate faster. Many algebraic properties can be used to help us calculate the answers for expressions faster. An example is the use of factorisation to calculate $2007^2 - 2006^2$. Do you know how to find the answer?



Expansion of Algebraic Expressions

In Book 1, we use the distributive law to simplify expressions like

$$2 \times (3 + 4) = 2 \times 3 + 2 \times 4$$
$$= 6 + 8 = 14 \qquad \text{and}$$

$$5 \times (9 - 6) = 5 \times 9 - 5 \times 6$$
$$= 45 - 30 = 15$$

i.e. multiplication is distributive over addition and subtraction.

Similarly we can use the distributive laws to simplify algebraic expressions when we remove the brackets:

$$3 \times (a + b) = 3 \times a + 3 \times b = 3a + 3b$$

$$5 \times (x - y) = 5 \times x + 5 \times (-y) = 5x - 5y$$

$$2a \times (3b + c) = 2a \times 3b + 2a \times c = 6ab + 2ac$$

$$4x \times (5y - 2z) = 4x \times 5y + 4x \times (-2z) = 20xy - 8xz$$

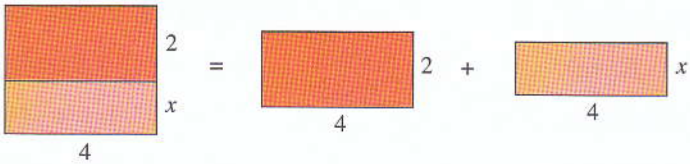
$$-2h \times (4h + 5k) = (-2h) \times 4h + (-2h) \times 5k$$
$$= -8h^2 - 10hk$$

$$-3m \times (6m - 5n) = (-3m) \times 6m + (-3m) \times (-5n)$$
$$= -18m^2 + 15mn$$

Take note that the rules for algebraic multiplication are the same as those for arithmetic multiplication, i.e.

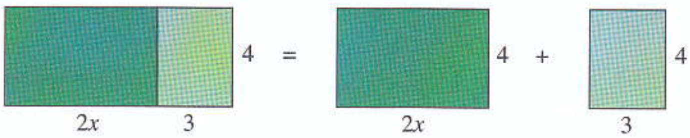
$(+3) \times (+4) = 12$	as	$(+2a) \times (+c) = +2ac$
$(+3) \times (-4) = -12$	as	$(+4x) \times (-2z) = -8xz$
$(-3) \times (+4) = -12$	as	$(-2h) \times (+5k) = -10hk$
$(-3) \times (-4) = 12$	as	$(-3m) \times (-5n) = 15mn$

The distributive law of multiplication over addition can also be illustrated by using the idea of the area of a rectangle.



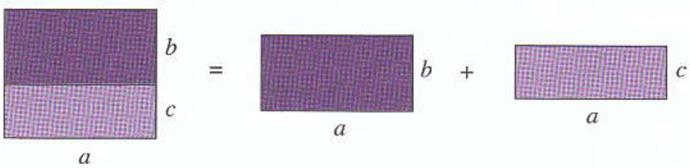
i.e. $4 \times (2 + x) = 4 \times 2 + 4 \times x$

and



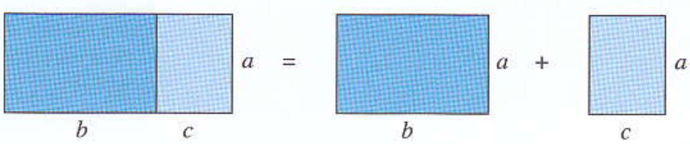
i.e. $(2x + 3) \times 4 = 2x \times 4 + 3 \times 4$.

Similarly,



$a \times (b + c) = a \times b + a \times c$

and



$(b + c) \times a = b \times a + c \times a$.

You can find out the age of your friend without being told by asking your friend to carry out the following instructions in the given order:

1. *Write down your age.*
2. *Add 5 to it.*
3. *Double the result.*
4. *Add 10 to it.*
5. *Multiply the result by 5.*
6. *Subtract 100 from it.*

Ask your friend to tell you the final result. Then cross out the last digit to obtain your friend's age. Try this mathematical game with different people. Do you always get their ages right? If so, can you explain why?

Example 1

Expand and simplify.

(a) $2(2x + 3) + 5(x - 4)$
(b) $-3(2x + 4) - 2(x - 5)$

Solution

(a) $2(2x + 3) + 5(x - 4)$
$= 2(2x) + 2(3) + 5(x) + 5(-4)$
$= 4x + 6 + 5x - 20$
$= 9x - 14$

(b) $-3(2x + 4) - 2(x - 5)$
$= -3 \times 2x + (-3) \times 4 - 2 \times x - 2 \times (-5)$
$= -6x - 12 - 2x + 10$
$= -8x - 2$

Example 2

Expand and simplify.

(a) $2x(x + 2y) + 3x(2x - 3y)$
(b) $-3a(h + k) - 4a(2h - 5k)$

Solution

(a) $2x(x + 2y) + 3x(2x - 3y)$
$= 2x \times x + 2x \times (2y) + 3x \times (2x) + 3x \times (-3y)$
$= 2x^2 + 4xy + 6x^2 - 9xy$
$= 8x^2 - 5xy$

Notes

You can skip the first step once you become familiar with it.

(b) $-3a(h + k) - 4a(2h - 5k)$
$= -3a \times h + (-3a) \times k - 4a \times 2h - 4a \times (-5k)$
$= -3ah - 3ak - 8ah + 20ak$
$= -11ah + 17ak$

Exercise 3a

1. Expand each of the following:

(a) $3(2a + b)$

(b) $4(3x + 4y)$

(c) $5(3c - 4d)$

(d) $7(5h - 2k)$

(e) $-3(2h + 7k)$

(f) $-8(3a + 5b)$

(g) $-6(7x - 3y)$

(h) $-9(3h - 2k)$

(i) $-2(3 + 4x)$

(j) $-5(2 - 3x)$

2. Expand each of the following:

(a) $2x(5x + y)$

(b) $3a(2a + 7b)$

(c) $6x(3x - y)$

(d) $5a(a - b)$

(e) $-x(5h + 3k)$

(f) $-3a(2a + 3b)$

(g) $-2h(2h - 3k)$

(h) $-4m(2m - 5n)$

(i) $-7a(-2a + 3b)$

(j) $-3k(-2h - 7k)$

3. Expand and simplify each of the following:

(a) $2(x + 1) + 3(x + 5)$

(b) $4(2x+ 3) + 5(x + 3)$

(c) $5(h + 3) - 2(4 + h)$

(d) $7(2x + 1) - 4(x + 3)$

(e) $8(3h - 4) + 5(h - 2)$

(f) $6(5x - 4) + 2(3x - 2)$

(g) $3(2h - 1) - 2(5h - 3)$

(h) $4(3x - 5) - 6(2 - 3x)$

(i) $9(5 - 2x) + 3(6 - 5x)$

(j) $7(3x - 4) + 4(3 - 2x)$

4. Expand and simplify each of the following:

(a) $x(2x + 1) + x(x + 4)$

(b) $x(3x + 1) + 2x(x + 3)$

(c) $2x(2x + 3) - x(2 - 5x)$

(d) $3x(5 + x) - 2x(3x - 7)$

(e) $-2x(3 + 4x) - 5x(x - 1)$

(f) $p(5p - q) - 2p(3q - p)$

(g) $3a(2a - b) + 3a(b - 3a)$

(h) $2a(b - 2a) - 4a(a - 2b)$

(i) $2p(3p - 5q) - p(2q - p)$

(j) $4x(3x + y) - 3x(2x - 5y)$

Further Algebraic Expansions

We will now learn to expand the product of two expressions with two terms each and the product of an expression with two terms and an expression with three terms.

Example 3

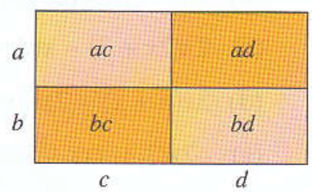
Expand each of the following

(a) $(a + b)(c + d)$
(b) $(a + b)(c + d + e)$

Solution

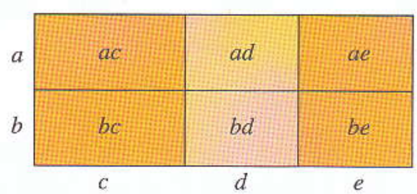
Method 1
We may use the idea of area of rectangles to illustrate the concept of the product of two algebraic expressions.

(a)



$(a + b)(c + d) = ac + ad + bc + bd$ shows that the area of the rectangle whose sides are $(a + b)$ and $(c + d)$ is equal to the sum of the areas of the four smaller rectangles.

(b) Similarly,



i.e. $(a + b)(c + d + e) = ac + ad + ae + bc + bd + be$

Method 2

(a) Let $c + d = k$.

Now $(a + b)(c + d) = (a + b)k$

$= ak + bk$ (Distributive property)

$= a(c + d) + b(c + d)$ (Replace k by $c + d$)

$= ac + ad + bc + bd$ (Distributive property)

(b) Let $(c + d + e) = k$.

Now $(a + b)(c + d + e) = (a + b)k$

$= ak + bk$ (Distributive property)

$= a(c + d + e) + b(c + d + e)$ (Replace k by $c + d + e$)

$= ac + ad + ae + bc + bd + be$

Method 3

Notice from Method 2 that in order to multiply two algebraic expressions, we multiply each term of one expression by each term of the other and simplify the result.

The above can be worked out more easily by using the following method.

(a) $(a + b)(c + d) = ac + ad + bc + bd$ (Multiply across each term)

(b) $(a + b)(c + d + e) = ac + ad + ae + bc + bd + be$

To do this procedure correctly, start with the first term in the first expression and multiply it with each term of the second expression:

$(a + b)(c + d)$
$= ac + ad + \ldots\ldots$

before doing the same with the second term in the first expression:

$(a + b)(c + d)$

$= ac + ad + bc + bd.$

Another way is to remember the acroynm FOIL, which stands for ***F****irst,* ***O****utside,* ***I****nside,* ***L****ast:*

$(a + b)(c + d)$

F*irst terms in each expression*

$= ac + \ldots\ldots$

$(a + b)(c + d)$

*Terms on the '****O****utside'*

$= ac + ad + \ldots\ldots$

$(a + b)(c + d)$

*Terms on the '****I****nside'*

$= ac + ad + bc + \ldots\ldots$

$(a + b)(c + d)$

L*ast terms in each expression*

$= ac + ad + bc + bd$

However, FOIL does not apply for product of expressions with more than two terms each.

Example 4

Expand the following:

(a) $(a + 3)(a + 4)$

(b) $(2d + 3e)(d + 2e)$

(c) $(x - 2y)(x + 5y)$

(d) $(2x - y)(3x + 2y + 1)$

Solution

(a) $(a + 3)(a + 4) = a^2 + 4a + 3a + 12$
$\quad = a^2 + 7a + 12$

(b) $(2d + 3e)(d + 2e) = 2d^2 + 4de + 3de + 6e^2$
$\quad = 2d^2 + 7de + 6e^2$

(c) $(x - 2y)(x + 5y) = x^2 + 5xy - 2xy - 10y^2$
$\quad = x^2 + 3xy - 10y^2$

(d) $(2x - y)(3x + 2y + 1) = 6x^2 + 4xy + 2x - 3xy - 2y^2 - y$
$\quad = 6x^2 + xy + 2x - 2y^2 - y$

Example 5

Simplify the following:

(a) $(x + 5)(x - 4) - (x + 2)(x - 3)$,

(b) $(x - 2y)(2x + y) - (3x + y)(5x - 4y)$.

Care must be taken with regard to signs.

Solution

(a) $(x + 5)(x - 4) - (x + 2)(x - 3)$
$= x^2 - 4x + 5x - 20 - (x^2 - 3x + 2x - 6)$ (Don't forget the brackets)
$= x^2 + x - 20 - x^2 + x + 6$
$= 2x - 14$

(b) $(x - 2y)(2x + y) - (3x + y)(5x - 4y)$
$= 2x^2 + xy - 4xy - 2y^2 - (15x^2 - 12xy + 5xy - 4y^2)$ (Don't forget the brackets)
$= 2x^2 - 3xy - 2y^2 - 15x^2 + 7xy + 4y^2$
$= -13x^2 + 4xy + 2y^2$

Example 6

Simplify the following:

(a) $(x^2 + 2)(x + 5)$
(b) $(2x^2 + 3x + 2)(5x - 4)$.

Solution

(a) $(x^2 + 2)(x + 5)$

$= x^3 + 5x^2 + 2x + 10$

(b) $(2x^2 + 3x + 2)(5x - 4)$

$= 10x^3 - 8x^2 + 15x^2 - 12x + 10x - 8$
$= 10x^3 + 7x^2 - 2x - 8$

Exercise 3b

1. Expand the following:

(a) $(a + 3)(a + 7)$
(b) $(a - 5)(a - 6)$
(c) $(c - 2)(c + 7)$
(d) $(x + 1)(x - 9)$
(e) $(n + 9)(n + 9)$
(f) $(n - 9)(n - 9)$
(g) $(x + 3y)(x - y)$
(h) $(b - 4c)(b + 4c)$
(i) $(m + 2p)(m - 3p)$
(j) $(e + f)(e + 5f)$

2. Simplify each of the following:

(a) $5 + (x + 1)(x + 3)$
(b) $3x + (2x - 1)(x + 7)$
(c) $6x^2 + (2x + 3)(x - 1)$
(d) $2x - (x - 4)(5x - 6)$
(e) $4x^2 - (3x - 4)(2x + 1)$
(f) $15 - (x - 4)(x - 7)$
(g) $(x + 2)(x + 1) - 2(x + 5)$
(h) $(x - 3)(x - 4) + 2x(x - 4)$
(i) $2x(3x - 4) - (x - 1)(x + 3)$
(j) $(2x - 1)(x + 5) - 5x(x - 4)$

3. Find the following products:

(a) $(x^2 + x + 1)(x + 2)$
(b) $(x^2 - x - 1)(x + 1)$
(c) $(x^2 + 2x - 1)(x - 1)$
(d) $(x^2 - 2x + 3)(x - 2)$
(e) $(a^2 + 3a - 2)(a + 3)$
(f) $(a^2 - 3a + 4)(a - 3)$
(g) $(x + 4)(x^2 - 5x + 7)$
(h) $(x - 5)(x^2 + 4x - 1)$
(i) $(x + 3)(x^2 - 7x - 2)$
(j) $(2x + 1)(x^2 + 3x - 1)$

Perfect Squares and Difference of Two Squares

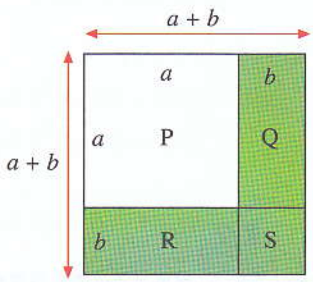
We will now learn three algebraic identities.

Example 7

Expand (a) $(a + b)(a + b)$, *(b)* $(a - b)(a - b)$, *(c)* $(a + b)(a - b)$.

Solution

(a) Consider the area of a square whose sides are $(a + b)$



The area of the square whose sides are $(a + b)$ is equal to the sum of the areas of P, Q, R and S.

$$(a + b)(a + b) = a^2 + ab + ab + b^2$$
$$= a^2 + 2ab + b^2$$
$$\therefore \quad (a + b)^2 = a^2 + 2ab + b^2$$

Your Attention

You cannot expand $(a + b)^2$ by bringing in the square like this:

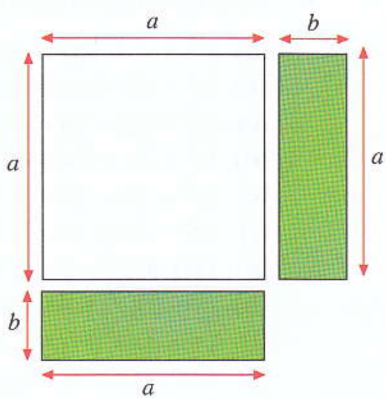
$$(a + b)^2 = a^2 + b^2$$

This is ***WRONG.*** *As you can see from the diagram, there are two extra rectangles Q and R, so $(a + b)^2 = a^2 + 2ab + b^2$.*

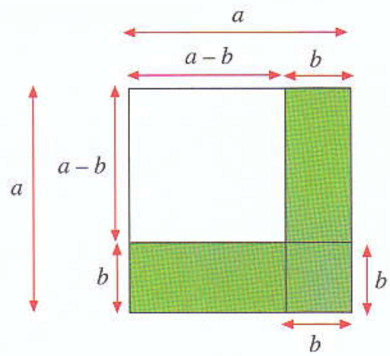
Notes

$(a + b)(a + b) = (a + b)^2$

(b) Consider the area of a square whose sides are a and the area of 2 rectangles whose length and width are a and b respectively.



The 2 rectangles are then placed on the square like this :



The area of the unshaded square is equal to $(a-b)(a-b)$

$\therefore \quad (a-b)(a-b) =$ Area of original square of sides a
– Area of 2 rectangles of sides a and b
+ Area of small square of sides b

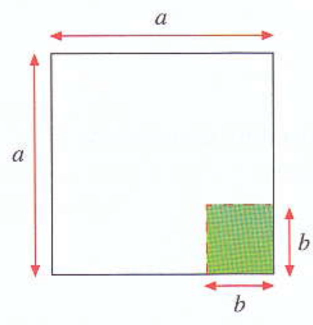
(because you subtract the area of the small square twice, you must add one area.)

$= a^2 - ab - ab + b^2$

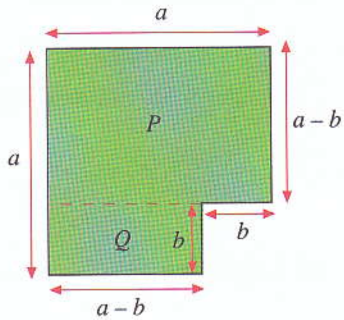
$= a^2 - 2ab + b^2$

$\therefore \quad (a-b)^2 = a^2 - 2ab + b^2$

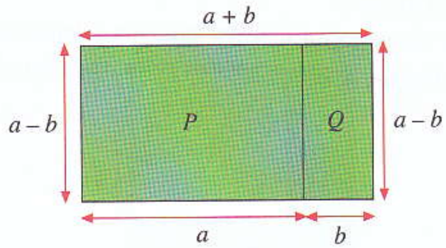
(c) Consider a square of sides a. A small square of sides b is to be cut out from the square of sides a.



After the square of sides b is cut out, we get a figure which is comprised of a rectangle P and a rectangle Q like this :



The rectangle Q is then cut out and placed next to rectangle P like this :



You cannot write $(a-b)(a-b) = a^2 - b^2$ because
$(a-b)(a-b)$
$= (a-b)^2$
$= a^2 - 2ab + b^2$
(see Example 7b solution). As you can see from the diagram, it is actually $(a+b)(a-b) = a^2 - b^2$
NOT
$(a-b)(a-b) = a^2 - b^2$.

The area of the shaded rectangle is equal to $(a + b)(a - b)$
= Area of rectangle P + Area of rectangle Q
$= a(a - b) + b(a - b)$
$= a^2 - ab + ab - b^2$
$= a^2 - b^2$
$\therefore \quad (a + b)(a - b) = a^2 - b^2$

The results of Example 7 are very useful in expanding expressions of a similar nature. These results are summarised below:

1. $\boldsymbol{(a + b)^2 = a^2 + 2ab + b^2}$
2. $\boldsymbol{(a - b)^2 = a^2 - 2ab + b^2}$
3. $\boldsymbol{(a + b)(a - b) = a^2 - b^2}$

They are all algebraic identities.

$(a + b)^2$ and $(a - b)^2$ are called **perfect squares** and $a^2 - b^2$ is called the **difference of two squares**.

Example 8

Expand the following expressions:

(a) $(2x + 3y)^2$,
(b) $(5x - 3y)^2$,
(c) $(3x + 2y)(3x - 2y)$.

Solution

(a) $(2x + 3y)^2 = (2x)^2 + 2(2x)(3y) + (3y)^2$ (Use Identity 1 where $a = 2x$ and $b = 3y$)
$= 4x^2 + 12xy + 9y^2$

(b) $(5x - 3y)^2 = (5x)^2 - 2(5x)(3y) + (3y)^2$ (Use Identity 2 where $a = 5x$ and $b = 3y$)
$= 25x^2 - 30xy + 9y^2$

(c) $(3x + 2y)(3x - 2y) = (3x)^2 - (2y)^2$ (Use Identity 3 where $a = 3x$ and $b = 2y$)
$= 9x^2 - 4y^2$

We can also use the technique learnt earlier in Example 4 to expand the above.

(a) $(2x + 3y)^2 = (2x + 3y)(2x + 3y)$
$= 4x^2 + 6xy + 6xy + 9y^2$
$= 4x^2 + 12xy + 9y^2$

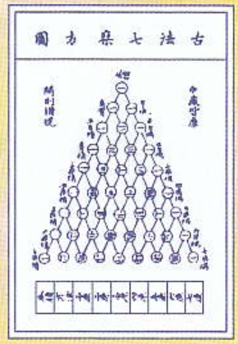
(b) $(5x - 3y)^2 = (5x - 3y)(5x - 3y)$
$= 25x^2 - 15xy - 15xy + 9y^2$
$= 25x^2 - 30xy + 9y^2$

(c) $(3x + 2y)(3x - 2y) = 9x^2 - 6xy + 6xy - 4y^2$
$= 9x^2 - 4y^2$

Which method do you like better?

FOR YOUR Information

The Chinese were familiar with expansions of $(a + b)^3$, $(a + b)^4$, $(a + b)^5$, etc. It appears that the famous "Pascal Triangle" was known in China long before Blaise Pascal was even born. The diagram below appeared in the 13th century text. "Detailed Analysis of the Mathematics Rules in the Nine Chapters".



Example 9

Use algebraic identities to evaluate the following:

(a) 201×199,
(b) 797^2,
(c) $39^2 + 78 + 1$.

Solution

It is harder to evaluate these expressions directly without using a calculator.

So we use the above identities to help.

(a) $201 \times 199 = (200 + 1)(200 - 1)$
$= 200^2 - 1^2$ (Using Identity 3)
$= 40\,000 - 1$
$= 39\,999$

(b) $797^2 = (800 - 3)^2$
$= 800^2 - 2(800)(3) + 3^2$ (Using Identity 2)
$= 640\,000 - 4800 + 9$
$= 635\,209$

(c) $39^2 + 78 + 1 = 39^2 + 2(39)(1) + 1^2$ (Using Identity 1)
$= (39 + 1)^2$
$= 40^2$
$= 1600$

Exercise 3c

1. Expand each of the following:
   (a) $(a + 4)^2$
   (b) $(m + 7)^2$
   (c) $(x - 9)^2$
   (d) $(2x - 1)^2$
   (e) $(1 - 3x)^2$
   (f) $(2 + 3k)^2$
   (g) $(p - 2q)^2$
   (h) $(c - 4d)^2$
   (i) $(5x - 3)^2$
   (j) $(4 - 3t)^2$

2. Expand each of the following:
   (a) $(2x - 5)(2x + 5)$
   (b) $(3x + 4)(3x - 4)$
   (c) $(5x - 2)(5x + 2)$
   (d) $(-2x + 3)(-2x - 3)$
   (e) $(7 - 2x)(7 + 2x)$
   (f) $(2h - k)(2h + k)$
   (g) $(6x + 5)(6x - 5)$
   (h) $(5p - q)(5p + q)$
   (i) $(2x - 11)(2x + 11)$
   (j) $(x^2 - y)(x^2 + y)$

3. Use the algebraic identities to evaluate each of the following:
   (a) 502×498
   (b) 305×295
   (c) 98×102
   (d) 1203^2
   (e) 901^2
   (f) 899^2
   (g) 892^2
   (h) 805^2
   (i) $69^2 + 138 + 1$
   (j) $78^2 + 312 + 4$

4. If $x^2 + y^2 = 14$ and $xy = 5$, find the value of $(x + y)^2$.

5. Given that $a + b = 10$ and $a^2 - b^2 = 40$, find the value of $a - b$.

6. If $x^2 + y^2 = 86$ and $xy = -16$, find the value of $(x - y)^2$.

7. If $(a + b)^2 = 361$ and $ab = -120$, calculate the value of $a^2 + b^2$.

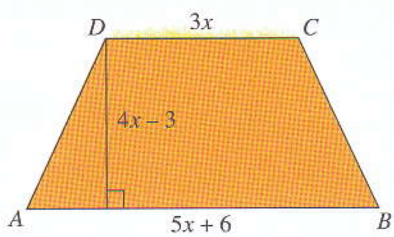
8. Using alegbraic rules and without using a calculator, evaluate

   (a) $\dfrac{236}{236^2 - 238 \times 234}$

   (b) $\dfrac{18 \times 164}{164^2 - 161 \times 167}$

   (c) $\dfrac{88\,888\,888}{(88\,888\,889)^2 - 88\,888\,888 \times 88\,888\,890}$

9. The figure shows $ABCD$ as a trapezium in which $AB = (5x + 6)$ cm, $DC = 3x$ cm and the height between the parallel lines is $(4x - 3)$ cm. Show that the area of the trapezium can be expressed as $(16x^2 - 9)$ cm^2.



Factorisation of Algebraic Expressions

In Book 1, we learnt how to factorise simple algebraic expressions by using the reverse of the distributive law as well as by grouping. We shall do a quick revision of the factorisation of these before we proceed to factorise quadratic expressions and solve quadratic equations.

Example 10

Factorise each of the following.

(a) $2 + 6x$

(b) $5x + 15y$

(c) $3x^2 + 9x$

(d) $2\pi r^2 + 2\pi rh$

(e) $2xy + 4yz - 8xyz$

(f) $a^2b^3 - a^5b^2 + 2a^4b^6$

Solution

(a) $2 + 6x = 2(1 + 3x)$

(b) $5x + 15y = 5(x + 3y)$

(c) $3x^2 + 9x = 3x(x + 3)$

(d) $2\pi r^2 + 2\pi rh = 2\pi r(r + h)$

(e) $2xy + 4yz - 8xyz = 2y(x + 2z - 4xz)$

(f) $a^2b^3 - a^5b^2 + 2a^4b^6 = a^2b^2(b - a^3 + 2a^2b^4)$

Notes

Identify the common factor for all the terms first.

Example 11

Factorise each of the following.

(a) $ab + 4a + 3b + 12$
(b) $x^2 + xy - 3x - 3y$
(c) $a^2 - ab - 2b + 2a$
(d) $x^3 - x^2 - 1 + x$
(e) $(a + 2b)^2 - (a + 2b)(3a - 7b)$

Solution

(a) $ab + 4a + 3b + 12 = (ab + 4a) + (3b + 12)$
$= a(b + 4) + 3(b + 4)$
$= (b + 4)(a + 3)$

(b) $x^2 + xy - 3x - 3y = (x^2 + xy) - (3x + 3y)$
$= x(x + y) - 3(x + y)$
$= (x + y)(x - 3)$

(c) $a^2 - ab - 2b + 2a = (a^2 - ab) + (2a - 2b)$
$= a(a - b) + 2(a - b)$
$= (a - b)(a + 2)$

Alternatively, $a^2 - ab - 2b + 2a = (a^2 - ab) - (2b - 2a)$
$= a(a - b) - 2(b - a)$
$= a(a - b) + 2(a - b)$
$= (a - b)(a + 2)$

Notes
$(b - a) = -(a - b)$

(d) $x^3 - x^2 - 1 + x = x^2(x - 1) - (1 - x)$
$= x^2(x - 1) + (x - 1)$
$= (x - 1)(x^2 + 1)$

(e) $(a + 2b)^2 - (a + 2b)(3a - 7b) = (a + 2b)[(a + 2b) - (3a - 7b)]$
$= (a + 2b)[a + 2b - 3a + 7b]$
$= (a + 2b)(9b - 2a)$

Exercise 3d

1. Factorise each of the following.
   - (a) $8x + 2$
   - (b) $4x - 20$
   - (c) $2x^2 + 6x$
   - (d) $5xy - 25yz$
   - (e) $3x^3 - 12x^2y$
   - (f) $10x^2 - 18x$
   - (g) $4xy - 12xyz$
   - (h) $14xy - 49y^2$
   - (i) $3a^2b + 6ab^2 + 3abc^2$
   - (j) $10x^3 + 15xy + 20xz$

2. Factorise each of the following.
   - (a) $x + xy + 2y + 2y^2$
   - (b) $x^2 + 3xy + 2x + 6y$
   - (c) $2ps + 2qs + qt + pt$
   - (d) $3sp + 3sq + 2tp + 2tq$
   - (e) $xz + xy + yz + y^2$
   - (f) $x^2 - 3x + 2xy - 6y$
   - (g) $6xy - 4x - 2z + 3yz$
   - (h) $2ac + bc - 2bd - 4ad$
   - (i) $xz - 3x - 3y + yz$
   - (j) $3bc - bd + 6ac - 2ad$

3. Factorise each of the following where possible.
   - (a) $1 + p^2 + p^3q + pq$
   - (b) $p^2q - pqr - 2pr + 2r^2$
   - (c) $49x^2 - 7x + 7ax - a$
   - (d) $3p^2 + 6pq - 4pr - 8qr$
   - (e) $4a^4 + 6a^3 + 2a^2 + 3a$
   - (f) $x^2y^2 - 5x^2y - 5xy^2 + xy^3$
   - (g) $5a^2 - a(2b - 30)$
   - (h) $4x^2 - 2x(a + b)$
   - (i) $m^3 + m^2(2m - 1)$
   - (j) $p(4m - n) - 2p(m + 2n)$

Factorisation Using Algebraic Identities

Earlier in this chapter, we found that

(1) $a^2 + 2ab + b^2 = (a + b)^2$,

(2) $a^2 - 2ab + b^2 = (a - b)^2$, and

(3) $a^2 - b^2 = (a + b)(a - b)$.

$(a + b)^2$ and $(a - b)^2$ are called **perfect squares** and $a^2 - b^2$ is called the **difference of two squares**. These identities are useful in helping us to factorise certain expressions.

> **Pay Attention**
> $(a + b)^2 \neq a^2 + b^2$
> $(a - b)^2 \neq a^2 - b^2$

Example 12

Factorise the following:

(a) $x^2 + 6x + 9$

(b) $t^2 - 12t + 36$

(c) $k^2 - 81$

(d) $9a^2 - 4$

Solution

(a) $x^2 + 6x + 9 = x^2 + 2(3)(x) + 3^2$
$= (x + 3)^2$

(b) $t^2 - 12t + 36 = t^2 - 2(6)(t) + 6^2$
$= (t - 6)^2$

(c) $k^2 - 81 = k^2 - 9^2$
$= (k + 9)(k - 9)$

(d) $9a^2 - 4 = (3a)^2 - 2^2$
$= (3a + 2)\ (3a - 2)$

Do it Yourself

Check your answers by expanding the factors and see whether you get back the original expressions.

Using factorisation can sometimes help us to do mental calculations in arithmetic (see next example).

Example 13

Evaluate the following by factorisation:

(a) $79 \times 83 - 69 \times 83$,

(b) $103^2 - 9$.

Solution

(a) $79 \times 83 - 69 \times 83 = 83(79 - 69)$

$= 83(10)$

$= 830$

(b) $103^2 - 9 = 103^2 - 3^2$

$= (103 + 3)(103 - 3)$

$= 106 \times 100$

$= 10\ 600$

Exercise 3e

1. Factorise each of the following completely.

(a) $x^2 + 8x + 16$
(b) $x^2 + 4x + 4$
(c) $a^2 + 6a + 9$
(d) $2x^2 + 4x + 2$
(e) $3x^2 + 12x + 12$
(f) $4x^2 + 32x + 64$
(g) $x^2 - 6x + 9$
(h) $x^2 - 8x + 16$
(i) $h^2 - 4h + 4$
(j) $2x^2 - 4x + 2$

2. Factorise each of the following completely.

(a) $x^2 + 2xy + y^2$
(b) $x^2 + 6xy + 9y^2$
(c) $x^2 + 8xy + y^2$
(d) $9a^2 + 24ab + 16b^2$
(e) $4x^2 + 8xy + 4y^2$
(f) $25x^2 - 10xy + y^2$
(g) $49y^2 - 42yz + 9z^2$

3. Factorise each of the following completely.

(a) $x^2 - 4$
(b) $x^2 - 16$
(c) $k^2 - 81$
(d) $25a^2 - 64$
(e) $36x^2 - 49$
(f) $81 - 16x^2$
(g) $64 - 9a^2$
(h) $-4h^2 + 81$
(i) $2x^2 - 18$
(j) $3x^2 - 147$

4. Factorise each of the following completely.

(a) $h^2 - k^2$
(b) $x^2 - 16y^2$
(c) $4c^2 - 25d^2$
(d) $36b^2 - a^2$
(e) $49c^2 - 9d^2$
(f) $2x^2 - 50y^2$
(g) $3x^2 - 27y^2$
(h) $64a^2 - 4b^2$
(i) $k^2 - \frac{1}{4}h^2$

5. Evaluate the following by factorisation.
   (a) $59^2 - 41^2$
   (b) $68^2 - 32^2$
   (c) $103^2 - 9$
   (d) $7.7^2 - 2.3^2$
   (e) $26.7^2 - 23.3^2$
   (f) $256^2 - 156^2$
   (g) $892^2 - 8^2$
   (h) $903^2 - 97^2$
   (i) $763^2 - 237^2$
   (j) $659^2 - 341^2$

6. Evaluate the following without using a calculator.
   (a) $36 \times 490 + 36 \times 51$
   (b) $5.16 \times 5.6 + 5.16 \times 4.4$
   (c) $27 \times 365 - 27 \times 265$
   (d) $587 \times 23 - 23 \times 487$
   (e) $395 \times 47 - 47 \times 385$
   (f) $84^2 - 84 \times 74$

*7. Factorise each of the following completely.
   (a) $x^2 - (y + 1)^2$
   (b) $c^2 - (d + 2)^2$
   (c) $(a + 3)^2 - 9$
   (d) $16 - 25(a + 3)^2$
   (e) $4(x + 1)^2 - 49$
   (f) $36 - 25(a + 1)^2$
   (g) $1 - 25x^2$
   (h) $49a^2 - (b + 5)^2$
   (i) $(2x - 1)^2 - 4y^2$
   (j) $25a^2 - (b - 1)^2$

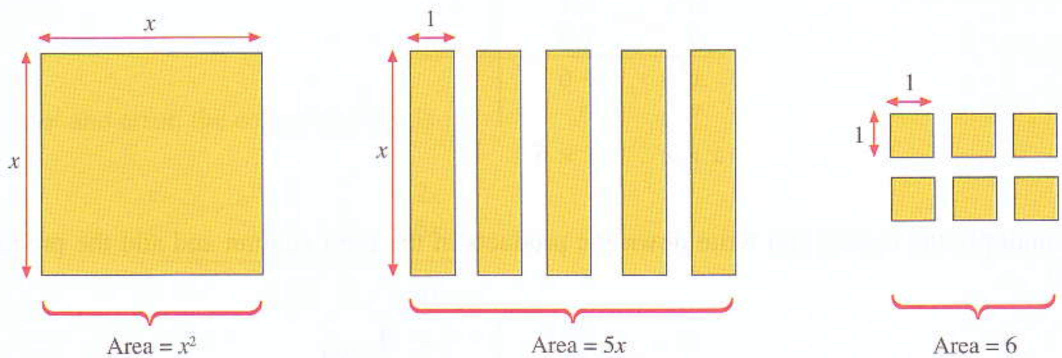
Factorisation of Quadratic Expressions

The general form of a quadratic expression is $ax^2 + bx + c$ where a, b and c are real numbers and $a \neq 0$. The expression has three terms: the term in x^2 is ax^2, the term in x is bx and the constant term is c.

Consider the quadratic expression $x^2 + 5x + 6$. To factorise a quadratic expression is to express it as a product of two factors, where each factor is not equal to 1.

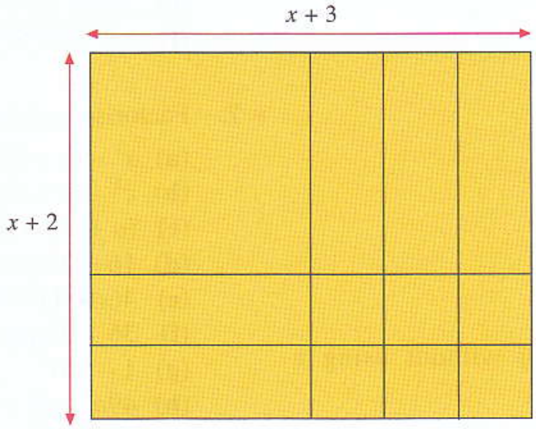
Method 1

We can represent $x^2 + 5x + 6$ by the **area** of the following square and rectangular tiles:



Can you arrange all these tiles to form a rectangle?

The following diagram shows one way of arranging all the tiles to form a rectangle.



Therefore, $x^2 + 5x + 6$ = area of all the tiles

= area of the big rectangle with length $x + 3$ and width $x + 2$

$= (x + 2)(x + 3)$

Method 2

Method 1 gives you a geometric understanding of what it means to factorise $x^2 + 5x + 6$. But it is too tedious. We will use a shortcut.

Notice that the 6 small square tiles are arranged in the form of a small 2×3 rectangle. This suggests that we have to find the factors of 6 :

$$6 = 1 \times 6$$
$$= 2 \times 3$$

Then, we try the first pair of factors as follows:

x	$+1$	
x	$+6$	
x^2	$+6$	
‖	‖	
$x \times x$	1×6	

Cross multiply the factors and write down the products in the third column and add the products as shown:

x	$+1$	$+x$ } add
x	$+6$	$+6x$ }
x^2	$+6$	$+7x \neq +5x$ $\therefore$ reject

If the final result is not equal to the term in x, then reject this and try the next pair of factors of 6, which are 2 and 3:

x ╲╱	$+2$	$+2x$ } add
x ╱╲	$+3$	$+3x$
x^2	$+6$ ‖ 2×3	$+5x$ $\therefore$ accept

Since the final result is equal to the term in x, we accept this solution:

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

We shall call this method the 'cross' method.

Notes

In this example, notice that :

$6 = 1 \times 6 \rightarrow 1 + 6 = 7$ ✗

$= 2 \times 3 \rightarrow 2 + 3 = 5$ ✓

*where the term in x is $5x$. So there is no need to try the 'cross' method for the first pair of factors of 6. This shortcut **only** works for $x^2 + bx + c$ where b, and $c > 0$.*

Your Attention

Remember to check your answer by expanding $(x + 2)(x + 3)$ to see if it will give $x^2 + 5x + 6$.

Example 14

Factorise $x^2 + 8x + 12$

Solution

$12 = 1 \times 12$
$\quad = 2 \times 6$
$\quad = 3 \times 4$

Using trial and error, the correct answer is :

x ╲╱	$+2$	$+2x$
x ╱╲	$+6$	$+6x$
x^2	$+12$	$+8x$

$$\therefore \quad x^2 + 8x + 12 = (x + 2)(x + 6)$$

Notes

$12 = 1 \times 12 \rightarrow 1 + 12 = 13$

$= 2 \times 6 \rightarrow 2 + 6 = 8$

$= 3 \times 4 \rightarrow 3 + 4 = 7$

$\therefore 2 + 6 = 8$ will give 8 (the term in x)

Example 15

Factorise $x^2 - 5x + 6$

Solution

$6 = 1 \times 6$ or $(-1) \times (-6)$

$\quad = 2 \times 3$ or $(-2) \times (-3)$

First trial :

x	$+2$	$+2x$
x	$+3$	$+3x$
x^2	$+6$	$+5x \neq -5x \quad \therefore$ reject

Notice that the term in x is $-5x$ (negative). So let's consider $6 = (-1) \times (-6)$.

Second trial :

x	-1	$-x$ } add
x	-6	$-6x$ } add
x^2	$+6$	$-7x \neq -5x \quad \therefore$ reject

Let's consider $6 = (-2) \times (-3)$

Third trial :

x	-2	$-2x$ } add
x	-3	$-3x$ } add
x^2	$+6$	$-5x \quad \therefore$ accept

Therefore, $x^2 - 5x + 6 = (x - 2)(x - 3)$.

Problem Solving Tips

For $ax^2 + bx + c$ where $a, c > 0$ but $b < 0$, both the factors of c must be negative, i.e.

$6 = (-1) \times (-6) \rightarrow$
$(-1) + (-6) = -7$ ✗
$6 = (-2) \times (-3) \rightarrow$
$(-2) + (-3) = -5$ ✓

This knowledge may help you to cut down unnecessary trial and error.

Notes

You do not have to show all the trials in your working.
If your first trial is correct, then just show the first trial.

Example 16

Factorise $x^2 + x - 6$

Solution

$-6 = 1 \times (-6)$ or $6 \times (-1)$
$\quad = 2 \times (-3)$ or $3 \times (-2)$

First trial

x	-1	$-x$
x	$+6$	$+6x$
x^2	-6	$+5x \neq +x \quad \therefore$ reject

Second trial

x	$+2$	$+2x$
x	-3	$-3x$
x^2	-6	$-x \neq +x \quad \therefore$ reject

Third trial

x	-2	$-2x$
x	$+3$	$+3x$
x^2	-6	$+x \therefore$ accept

Since the term in x is positive,the **bigger** factor of 6 (i.e. 3) must be **positive** and the smaller factor of 6 (i.e. 2) must be negative.

$\therefore \quad x^2 + x - 6 = (x-2)(x+3)$

Example 17

Factorise $x^2 - x - 6$

Solution

$-6 = (-1) \times 6$	or	$(-6) \times 1$
$= 2 \times (-3)$	or	$3 \times (-2)$

Using trial and error, the correct answer is:

x	$+2$	$+2x$
x	-3	$-3x$
x^2	-6	$-x$

$\therefore \; x^2 - x - 6 = (x + 2)(x - 3)$

Example 18

Factorise (a) $2x^2 + 7x + 3$
(b) $3x^2 - 17x + 20$

Solution

(a) The factors of $2x^2$ are $2x$ and x.

Both the term in x and the constant term are positive. So both factors of 3 must be positive.

First trial

$2x$	$+3$	$+3x$
x	$+1$	$+2x$
$2x^2$	$+3$	$+5x$
		(reject)

Second trial

$2x$	$+1$	$+x$
x	$+3$	$+6x$
$2x^2$	$+3$	$+7x$
		(accept)

$\therefore \; 2x^2 + 7x + 3 = (2x + 1)(x + 3)$

(b) The factors of $3x^2$ are $3x$ and x.

$20 = 1 \times 20$ or $(-1) \times (-20)$
$= 2 \times 10$ or $(-2) \times (-10)$
$= 4 \times 5$ or $(-4) \times (-5)$

First trial

$3x$	-20	$-20x$
x	-1	$-3x$
$3x^2$	$+20$	$-23x$

(reject)

Second trial

$3x$	-5	$-5x$
x	-4	$-12x$
$3x^2$	$+20$	$-17x$

(accept)

> **Notes**
> *Not all quadratic expressions can be factorised using the 'cross' method.*

$\therefore\ 3x^2 - 17x + 20 = (3x - 5)(x - 4)$

Example 19

Factorise each of the following.

(a) $x^2 - 16$ *(b)* $x^2 + 6x + 9$
(c) $px^2 - 4xp - 21p$ ∗*(d)* $2x^2y^2 + 5xy - 12$

Solution

(a) $x^2 - 16$

$x^2 \uparrow x \times x$, $16 \uparrow 4 \times (-4)$

$\therefore\ x^2 - 16 = (x + 4)(x - 4)$

x	$+4$	$4x$
x	-4	$-4x$
x^2	-16	$0x$

> **Notes**
> *Notice that there is no x term, i.e. $x^2 - 16 = x^2 + 0x - 16$. You can also use the algebraic identity $a^2 - b^2 = (a + b)(a - b)$ to factorise $x^2 - 16$.*

(b) $x^2 + 6x + 9$
$= (x + 3)(x + 3)$
$= (x + 3)^2$

x	$+3$	$3x$
x	$+3$	$3x$
x^2	9	$6x$

> **Notes**
> *You can also use the algebraic identity $a^2 + 2ab^2 + b^2 = (a + b)^2$ to factorise $x^2 + 6x + 9$.*

(c) $px^2 - 4xp - 21p$
$= p(x^2 - 4x - 21)$
$= p(x - 7)(x + 3)$

x	-7	$-7x$
x	$+3$	$3x$
x^2	-21	$-4x$

∗(d) $2x^2y^2 + 5xy - 12$
$= 2(xy)^2 + 5(xy) - 12$
$= (2xy - 3)(xy + 4)$

$2xy$	-3	$-3xy$
xy	$+4$	$8xy$
$2x^2y^2$	-12	$5xy$

Exercise 3f

1. Factorise each of the following expressions:
   (a) $x^2 + 6x + 8$
   (b) $y^2 + 3y + 2$
   (c) $m^2 + 9m + 8$
   (d) $b^2 + 11b + 28$
   (e) $x^2 - 11x + 24$
   (f) $e^2 - 4e + 4$
   (g) $m^2 - 9m + 20$
   (h) $x^2 + x - 2$
   (i) $a^2 - 9a + 14$
   (j) $a^2 + 2a - 8$

2. Factorise each of the following.
   (a) $2x^2 + 11x + 12$
   (b) $3a^2 + 10a + 7$
   (c) $4a^2 - 7a + 3$
   (d) $5p^2 - 13p + 6$
   (e) $6a^2 + 19a - 20$
   (f) $5p^2 + 7p - 6$
   (g) $6p^2 - 7p - 20$
   (h) $4a^2 - 7a + 3$
   (i) $4m^2 + 8m + 3$
   (j) $6p^2 + 7p - 20$

3. Factorise each of the following.
   (a) $4x^2 - 4x - 8$
   (b) $3x^2 + 15x + 18$
   (c) $4x^2 + 10x + 4$
   (d) $6x^2 + 15x - 36$
   (e) $8x^2 + 4x - 60$
   (f) $12x^2 + 10x + 2$
   (g) $18x^2 - 39x + 18$
   (h) $12x^2 + 10x - 12$
   (i) $4x^2 - 22x + 24$
   (j) $35x^2 + 55x - 30$

*4. Factorise the following expressions wherever possible:
   (a) $6a^2b^2 - 19ab - 20$
   (b) $6x^2y^2 + 5xy - 6$
   (c) $5p^2q^2 - 7pq - 6$
   (d) $6x^2y^2 - 7xy - 20$
   (e) $16 + 8xy + x^2y^2$
   (f) $25 - 10hk + h^2k^2$
   (g) $7hk - 15 + 2h^2k^2$
   (h) $3 - 8mn + 4m^2n^2$
   (i) $12p^2q^2 - 40 + 14pq$
   (j) $13hk + 6 + 5h^2k^2$

Solving Quadratic Equations by Factorisation

In arithmetic, we learnt that the product of any number and zero is equal to zero. For example $5 \times 0 = 0$, $0 \times 8 = 0$, $-6 \times 0 = 0$, $0 \times (-7) = 0$, $0 \times 0 = 0$ etc.

Similarly, in algebra, if two factors P and Q are such that $P \times Q = 0$, then either $P = 0$ or $Q = 0$ or both P and Q are equal to 0. We shall use this principle to solve quadratic equations.

The general form of a quadratic equation is $ax^2 + bx + c = 0$, where a, b and c are real numbers and $a \neq 0$.

Example 20

Solve the following equations.

(a) $x(x-3) = 0$

(b) $3a(2a+3) = 0$

(c) $(x+2)(x-3) = 0$

(d) $(2x-3)(4x-5) = 0$

Just for fun

Assume $x = y$, multiply both sides by x: $x^2 = xy$. Subtract y^2 from both sides:

$$x^2 - y^2 = xy - y^2$$

Factorise both sides:

$$(x+y)(x-y) = y(x-y)$$

Divide both sides by $x - y$:

$$x + y = y$$

But $x = y$,

$$2y = y$$

Divide both sides by y:

$\therefore\ 2 = 1$

Is it true that $2 = 1$?

Can you identify the mistake in the steps above?

Solution

(a) $x(x-3) = 0$

Either $x = 0$ or $x - 3 = 0$

$\therefore\ x = 0$ or $x = 3$

(b) $3a(2a+3) = 0$

Either $3a = 0$ or $2a + 3 = 0$

$\therefore\ a = 0$ or $2a = -3$

$a = -1\frac{1}{2}$

(c) $(x+2)(x-3) = 0$

Either $(x+2) = 0$ or $x - 3 = 0$

$\therefore\ x = -2$ or $x = 3$

(d) $(2x-3)(4x-5) = 0$

Either $(2x-3) = 0$ or $4x - 5 = 0$

$\therefore\ 2x = 3$ or $4x = 5$

$\therefore\ x = 1\frac{1}{2}$ or $x = 1\frac{1}{4}$

Example 21

Solve the following equations.

(a) $12x^2 - 4x = 0$ (b) $9x^2 - 4 = 0$

(c) $x^2 - 3x - 28 = 0$ (d) $2x^2 - 7x + 6 = 0$

Solution

(a)

$$12x^2 - 4x = 0$$
$$4x(3x - 1) = 0$$
$$\therefore \quad 4x = 0 \quad \text{or} \quad 3x - 1 = 0$$
$$\therefore \quad x = 0 \quad \text{or} \quad x = \frac{1}{3}$$

(b)

$$9x^2 - 4 = 0$$
$$(3x)^2 - 2^2 = 0$$
$$(3x + 2)(3x - 2) = 0$$
$$\therefore \quad 3x + 2 = 0 \quad \text{or} \quad 3x - 2 = 0$$
$$\therefore \quad x = -\frac{2}{3} \quad \text{or} \quad x = \frac{2}{3}$$

$3x$	$+2$	$6x$
$3x$	-2	$-6x$
$9x^2$	-4	0

Notes

We can also use the algebraic identity $a^2 - b^2 = (a + b)(a - b)$.

(c)

$$x^2 - 3x - 28 = 0$$
$$(x - 7)(x + 4) = 0$$
$$x - 7 = 0 \quad \text{or} \quad x + 4 = 0$$
$$\therefore \quad x = 7 \quad \text{or} \quad x = -4$$

x	-7	$-7x$
x	$+4$	$4x$
x^2	-28	$-3x$

(d)

$$2x^2 - 7x + 6 = 0$$
$$(2x - 3)(x - 2) = 0$$
$$\therefore (2x - 3) = 0 \quad \text{or} \quad x - 2 = 0$$
$$\therefore \quad x = 1\frac{1}{2} \quad \text{or} \quad x = 2$$

$2x$	-3	$-3x$
x	-2	$-4x$
$2x^2$	$+6$	$-7x$

Exercise 3g

1. Solve the following equations.
   (a) $x(x-9)=0$
   (b) $a(a+7)=0$
   (c) $2k(k-5)=0$
   (d) $5y(2y+1)=0$
   (e) $3h(5-4h)=0$
   (f) $3m(7+4m)=0$
   (g) $(x+2)(x+3)=0$
   (h) $(x+5)(x-7)=0$
   (i) $(k-4)(k+11)=0$
   (j) $(n-4)(n-9)=0$

2. Solve the following equations.
   (a) $x^2+9x=0$
   (b) $k^2-7k=0$
   (c) $2x^2+8x=0$
   (d) $5x^2+25x=0$
   (e) $3x^2-4x=0$
   (f) $3d-81d^2=0$
   (g) $4a^2-16a=0$
   (h) $5x^2-15x=0$
   (i) $7x^3+21x^2=0$

3. Solve the following equations.
   (a) $b^2-16=0$
   (b) $4n^2-25=0$
   (c) $64-a^2=0$
   (d) $3x^2-3=0$
   (e) $2e^2-50=0$
   (f) $4p^2-100=0$
   (g) $m^2-\frac{1}{4}=0$
   (h) $d^2-\frac{16}{25}=0$
   (i) $\frac{4}{9}-\frac{x^2}{25}=0$

4. Solve the following equations.
   (a) $e^2-16e+64=0$
   (b) $d^2+6d-27=0$
   (c) $a^2+12a+36=0$
   (d) $q^2+7q=60$
   (e) $b^2-7b-120=0$
   (f) $1+3a=10a^2$
   (g) $k^2-2k=63$
   (h) $3p^2-10p+8=0$
   (i) $2m^2+5m-3=0$

5. If $x=3$ is a solution of the equation $x^2+kx+15=0$, find the value of k. Hence find the other solution of the equation.

6. If $x=5$ is a solution of the equation $x^2-hx+10=0$, find the value of h. Hence find the other solution of the equation.

7. If $x=3$ is a solution of the equation $2x^2-5x+c=0$, find the value of c. Hence find the other solution of the equation.

Problem Solving Involving Quadratic Equations

Many mathematical and real-life problems can be solved with the help of quadratic equations.

Example 22

Find two consecutive positive odd numbers such that the sum of their squares is equal to 130.

Solution

Strategy 1: Make a systematic list

The odd numbers are: 1, 3, 5, 7, 9, 11, 13, ...

Their squares are: 1, 9, 25, 49, 81,121, 169, ...

From the above list, we have 49 + 81 = 130

$\therefore$ the two consecutive positive odd numbers are 7 and 9.

Strategy 2: Use an equation

Let one number be x. The next consecutive odd number will be $x + 2$.

Hence

$$\begin{aligned} x^2 + (x+2)^2 &= 130 \\ x^2 + x^2 + 4x + 4 &= 130 \\ 2x^2 + 4x - 126 &= 0 \\ x^2 + 2x - 63 &= 0 \\ (x-7)(x+9) &= 0 \end{aligned}$$

$\therefore \quad x = 7 \quad$ or $\quad x = -9$ (rejected because x is positive)

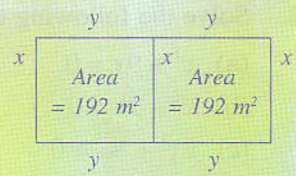
When $x = 7$, $x + 2 = 9$

$\therefore$ the two consecutive positive odd numbers are 7 and 9.

Check: $7^2 + 9^2 = 49 + 81 = 130$.

Just for Fun

Brian intends to construct the walls of two toy semi-detached houses (as shown below) using his 1-cm square toy building blocks. Each of the houses has an area of 192 cm².



Can you help him select the cheapest design to build (this will be the one with the smallest wall length, i.e., $3x + 4y$)?

Copy and complete the table to get the answer.

x	y	$3x + 4y$
192	1	580
96	2	
48		
24		
16		
12		
6		
3		
2		
1		

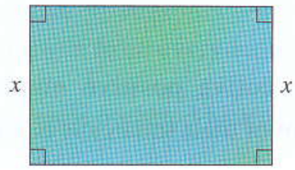
Which design is the cheapest to build? Explain why.

Example 23

The perimeter of a rectangle is 20 cm and its area is 24 cm². Calculate the length and breadth of the rectangle.

Solution

Let the breadth of the rectangle be x cm.



$\therefore$ the length of the rectangle $= \dfrac{20 - 2x}{2}$

$= 10 - x$ cm

$\therefore$ area of the rectangle $= x(10 - x) = 24$

$$10x - x^2 = 24$$

$$x^2 - 10x + 24 = 0$$

$$(x - 4)(x - 6) = 0$$

$$\therefore \quad x = 4 \text{ or } x = 6$$

When $x = 4$, length $= 10 - 4 = 6$ cm

When $x = 6$, length $= 10 - 6 = 4$ cm

Since we normally assign the longer side to length, the length = 6 cm and the breadth = 4 cm.

Check: When the length = 6 cm and the breadth = 4 cm,

its perimeter $= 2(6 + 4)$ cm $= 20$ cm and

its area $= (6 \times 4)$ cm^2 $= 24$ cm^2.

Exercise 3h

1. Find the whole number such that four times the number subtracted from three times the square of the number makes 15.

2. Find the whole number such that twice its square added to itself makes 10.

3. Find two consecutive positive numbers such that the sum of their squares is equal to 113.

4. Find two consecutive positive odd numbers such that the sum of their squares is 74.

5. Find two consecutive positive even numbers such that the sum of their squares is 164.

6. The difference between two numbers is 9 and the product of the numbers is 162. Find the two numbers.

7. A rectangular field, 70 m long and 50 m wide, has a path of uniform width around it. If the area of the path is 1024 m^2, find the width of the path.

8. The base and height of a triangle are $(x + 3)$ cm and $(2x - 5)$ cm respectively. If the area of the triangle is 20 cm^2, find x.

9. The difference between two numbers is 3. If the square of the smaller number is equal to 4 times the larger number, find the numbers.

10. The length of a rectangle is 5 cm longer than its width and its area is 66 cm^2. Find the perimeter of the rectangle.

11. Two positive numbers differ by 7 and the sum of their squares is 169. Find the numbers.

12. Two positive numbers differ by 5 and the square of their sum is 169. Find the numbers.

13. A piece of wire 44 cm long is cut into two parts and each part is bent to form a square. If the total area of the two squares is 65 cm^2, find the perimeter of the two squares.

14. A particle is projected from ground level so that its height above the ground after t seconds is given by $20t - 5t^2$ m. After how many seconds is it 15 m above the ground? Can you explain briefly why there are two possible answers?

SUMMARY

1. Algebraic identities :
   (a) $(a + b)^2 = a^2 + 2ab + b^2$
   (b) $(a - b)^2 = a^2 - 2ab + b^2$
   (c) $(a + b)(a - b) = a^2 - b^2$

2. Factorisation of algebraic expression can be done by
   (a) identifying and taking out all the common factors from every term in the given expression;
   (b) grouping terms in such a way that the new groups obtained have some common factors;
   (c) using the 'cross' method for quadratic expression.

3. If two factors P and Q are such that $P \times Q = 0$, then either $P = 0$, or $Q = 0$, or both P and Q are equal to 0. This principle is used to solve quadratic equations.

Review Examples 3

Example 1

Solve the following equations.
(a) $2x^2 - 32 = 0$
(b) $(x + 2)^2 = 9$
(c) $2x^2 + 5x = 0$
(d) $2x^2 + 5x - 3 = 0$

Solution

(a) $2x^2 - 32 = 0$
$2(x^2 - 16) = 0$
$2(x^2 - 4^2) = 0$
$2(x + 4)(x - 4) = 0$
$\therefore\ x + 4 = 0$ or $x - 4 = 0$
$x = -4$ or $x = 4$

(b) $(x + 2)^2 = 9$

$(x + 2)^2 - 9 = 0$

$(x + 2)^2 - 3^2 = 0$

$(x + 2 - 3)(x + 2 + 3) = 0$

$(x - 1)(x + 5) = 0$

$\therefore \quad x = 1 \quad \text{or} \quad x = -5$

(c) $2x^2 + 5x = 0$

$x(2x + 5) = 0$

$\therefore \quad x = 0 \quad \text{or} \quad 2x + 5 = 0$

$x = 0 \quad \text{or} \quad x = -2\frac{1}{2}$

(d) $2x^2 + 5x - 3 = 0$

$(2x - 1)(x + 3) = 0$

$\therefore \quad 2x - 1 = 0 \quad \text{or} \quad x + 3 = 0$

$x = \frac{1}{2} \quad \text{or} \quad x = -3$

$2x$	-1	$-x$
x	$+3$	$+6x$
$2x^2$	-3	$5x$

Example 2

A man is now 5 times as old as his son. Four years ago, the product of their ages was 52. Find their present ages.

Solution

Let the boy be x years old now.

Therefore his father is $5x$ years old.

4 years ago, their ages were $(x - 4)$ and $(5x - 4)$ respectively.

Hence $(x - 4)(5x - 4) = 52$

$5x^2 - 24x + 16 = 52$

$5x^2 - 24x - 36 = 0$

$(5x + 6)(x - 6) = 0$

$\therefore \quad x = -\frac{6}{5} \quad \text{or} \quad x = 6$

Since the boy cannot be $-\frac{6}{5}$ years old, the boy must now be 6 years old and his father 30 years old.

Check: 4 years ago, the boy was 2 years old and his father 26 years old. $(2 \times 26 = 52)$.

Just for Fun

Paul and Julie had a date one Saturday. They agreed to meet at Marina Square at 8 p.m. Julie thought that her watch was faster by 5 minutes but in actual fact, it was slower by 5 minutes. Paul thought that his watch was slower by 5 minutes but in actual fact, it was faster by 5 minutes. Julie deliberately turned up 10 minutes late while Paul turned up 10 minutes earlier. Who turned up first and how long did she/he have to wait for the other person?

Review Questions 3

1. Expand the following expressions:
   (a) $(2a + 3b)(3a + 4b)$
   (b) $(3a - 5b)(4a - b)$
   (c) $(5x + 2y)(x - 3y)$
   (d) $(7x - 4y)(x + 3y)$
   (e) $\left(\frac{2}{3}xy - 3\right)^2$
   (f) $\left(3a + \frac{4}{5}b\right)^2$
   (g) $\left(-\frac{1}{4}a - \frac{1}{6}b\right)^2$
   (h) $\left(\frac{1}{4}abc - \frac{3}{2}x^2yz\right)^2$
   (i) $\left(\frac{3}{4}xy + \frac{1}{3}ab\right)\left(\frac{3}{4}xy - \frac{1}{3}ab\right)$
   (j) $\left(\frac{x}{2} + \frac{y}{4}\right)\left(\frac{x}{2} - \frac{y}{4}\right)$

2. Factorise the following:
   (a) $4x - 8(y - 2z)$
   (b) $5a^2 + 10a(b + c)$
   (c) $(x + y)(a + b) - (y + z)(a + b)$
   (d) $(2x + y)^2 - 3(2x + y)$
   (e) $(2a - 3b)(p + q) + (a - b)(p + q)$
   (f) $5(m - 2n) - (m - 2n)^2$
   (g) $ax + by - ay - bx$
   (h) $x^2 - 2xy + xz - 2yz$
   (i) $3a^3 - 2a^2 + 3a - 2$
   (j) $x^3 + x^2 - 4x - 4$

3. Factorise the following completely:
   (a) $x^2 + 5x + 25$
   (b) $y^2 + 14y + 49$
   (c) $2z^2 + 12z + 18$
   (d) $x^2 - 8x + 16$
   (e) $a^2 - 12a + 36$
   (f) $3b^2 - 6b + 3$
   * (g) $6p^4 - 24q^2$
   * (h) $2p^4 - 18p^2q^2$
   * (i) $x^4y^2 - 4x^2y^4$
   * (j) $32xy^4 - 2x^5$
   * (k) $64a^4 - 4b^4$

4. Factorise the following:
   (a) $x^2 + 13x + 36$
   (b) $a^2 - 20a + 19$
   (c) $a^2 + 15a - 16$
   (d) $4p^2 - 12p + 9$
   * (e) $25p^2q^2 + 10pq + 1$
   (f) $9 + 6y + y^2$
   * (g) $1 + 12xy + 36x^2y^2$
   (h) $49 - 4a^2$
   * (i) $90x^2y^2 - 10$

5. Solve the following equations:
   (a) $12x - 20 = x^2$
   (b) $11x^2 = 26x + 21$
   (c) $12x + 9 = 5x^2$
   (d) $2x^2 - 11x + 5 = 0$
   (e) $3x^2 - 6x = 0$
   (f) $y^2 + 3y = 0$
   (g) $x^2 - 9 = 0$
   (h) $2b^2 - 8 = 0$
   (i) $(x + 3)^2 = 4$
   (j) $(a - 4)^2 = 25$

* 6. Solve the equation $9x^2y^2 - 12xy + 4 = 0$, giving y in terms of x.

7. The length and breadth of a rectangle are $(5x + 3)$ cm and $(3x - 2)$ cm. Write down in terms of x,
   (a) the perimeter,
   (b) the area.
   If the area of the rectangle is 230 cm^2, find the value of x and hence write down the perimeter of the rectangle.

8. A man walks for x hours at a speed of $(x + 1)$ km/h and cycles for $(x - 1)$ hours at a speed of $(2x + 5)$ km/h. If the total distance travelled is 90 km, find x.

9. A man is x^2 years old while his son is x years old. In $4x$ years time, the man will be twice as old as his son. Form an equation and solve it to find the value of x.

10. x is a number such that when $(x + 1)^2$ is divided by $(x - 2)$, the quotient is 16 and the remainder is $(x - 3)$. What are the values of x?

11. If each pupil in a class sends a New Year greeting card to every classmate, the total number of cards sent out will be 870. Find the number of pupils in the class.

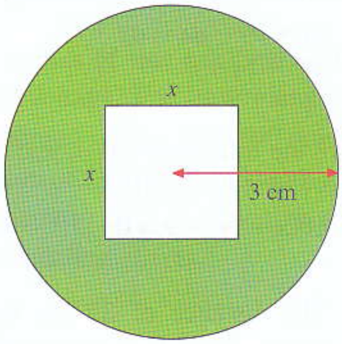
12. A cyclist travels 40 km from P to Q at an average speed of x km/h.
   (a) Write down, in terms of x, the time taken to travel from P to Q.
   (b) On the return journey from Q to P, the cyclist decreases his average speed by 3 km/h. Find the time taken for the return journey in terms of x.
   ***(c)** The difference between the time taken in **(a)** and **(b)** is 40 minutes.
   (i) Write down an equation in x and show that it reduces to $x^2 - 3x - 180 = 0$.
   (ii) Solve this equation to find the time taken to travel from P to Q.

13. The cost of hiring a bus for an outing was \$240 and this was to be shared equally by x people. On the day of the excursion, 4 people were unable to make it. The remaining people had to share the cost of hiring the bus instead, so that each person had to pay an extra \$2.

   Form an equation in x and solve it to find the amount that each person had to pay originally.

14. Mr Chan drove from his home to his office at an average speed of x km/h. The distance between his home and his office is 80 km.
   (a) Write down an expression, in terms of x, for the number of hours he took to drive from his home to his office.
   (b) When he drove from his office to his home, his average speed was 5 km/h faster than his speed when he drove from his home to his office.
   Write down an expression, in terms of x, for
   (i) his speed when he drove from his office to his home,
   (ii) the number of hours he took to drive from his office to his home.
   ***(c)** Given that Mr Chen took 15 minutes less to drive from his office to his home than from his home to his office, form an equation in x and show that it simplifies to
   $$x^2 + 5x - 1600 = 0$$
   (d) Solve the equation $x^2 + 5x - 1600 = 0$, giving both answers correct to three significant figures.
   (e) Calculate, correct to the nearest minute, the time he spent on driving from his home to his office.

15.



The diagram shows an ancient coin which was once used in China. The coin is a circle of 3 cm with a square of x cm removed from its centre.

(a) Find an expression for the area of one side of the coin in terms of x and π.

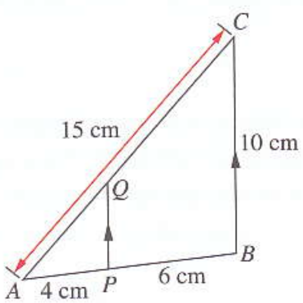
(b) The area of one side of the coin is 7π cm^2. Find the area of the square by forming an equation in x.
(Take π to be 3.142)

Revision Exercise I No. 1

1. (a) What is the largest number of books at \$3.50 each that you can buy with \$30? How much will you have left over?
   (b) An iron bar can be cut into 16 pieces, each 15 cm long. If 12-cm pieces were required instead, how many pieces could be cut from the bar?

2. A map is drawn using a scale of 1 cm to 8 km.
   (a) Find the R. F. of the map.
   (b) Find in cm, the distance on the map between two places which are 72 km apart.
   (c) Find in cm^2, the area on the map of a forest which has an area of 496 km^2.

3. (a) In the figure, $\triangle APQ$ is similar to $\triangle ABC$. Given that $AP = 4$ cm, $PB = 6$ cm, $BC = 10$ cm and $AC = 15$ cm, find the lengths of AQ and PQ.



   (b) Find the number of sides of a regular polygon whose exterior angle is 18°.

4. Simplify each of the following:
   (a) $2(5x - 2y) + 5(y - 3x)$
   (b) $3x(2x + y) - x(5x - 3y)$
   (c) $7x(x - y + 2z) - 5x(y - 3x - z)$

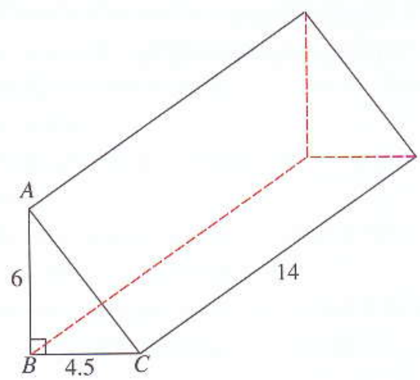
5. (a) If $5x - 8y = 3(x - y)$, find the numerical value of $\frac{x}{3y}$.
   (b) If $x^2 + y^2 = 57$ and $xy = 3$, find the value of $3(x + y)^2$.

6. Factorise each of the following:
   (a) $px + py - qx - qy$
   (b) $2c^2 - 5c + 3$

7. Solve the following equations:
   (a) $2x^2 + 5x - 3 = 0$
   (b) $16(x - 1)^2 - 9 = 0$

8. (a) The distance of 25 km between two towns, P and Q, is represented by a line of 5 cm on a map. If the scale of the map is $1 : 5x$, find the value of x.
   (b) Three of the interior angles of a pentagon are 88°, 113° and 139° while the other two are $7x$° and $13x$°. Find x.

9. The figure shows a triangular bar of cross-section ABC. If the length of the bar is 14 cm, $AB = 6$ cm and $BC = 4.5$ cm, find the volume of the bar.



10. (a) Given that y is directly proportional to the positive square root of x and that $y = 4$ when $x = 9$, find
   (i) the value of y when $x = 64$,
   (ii) the value of x when $y = 7$.
   (b) If y is inversely proportional to $(2x^2 + 5)$ and that $y = 7$ when $x = 2$, find the value of
   (i) y when $x = 3$,
   (ii) x when $y = 5$.

Revision Exercise I No. 2

1. (a) 20 tonnes of rice can feed 300 soldiers for 72 days. For how many days can 32 tonnes of rice feed 540 soldiers?
   (b) A sum of \$360 is shared among 3 people in the ratio 2 : 3 : 7. Calculate the largest and smallest shares.

2. Simplify
   (a) $3x^2 - 5x - \{7x - [-2x^2 + x + 4(x - 2)]\}$,
   (b) $8x - [-(5x - 4) + 2(x - 1)]$.

3. (a) Out of 240 pupils, 78 prefer Literature, 94 prefer Geography and the rest prefer History. Draw a pie chart to illustrate this information and state the angles in each sector.
   (b) The interior angle of a regular polygon is 35 times its exterior angle. How many sides has the polygon?

4. Simplify the following expressions:
   (a) $(2a + 5b)^2 - (a + 3b)(a - 6b)$
   (b) $(4a + b)(4a - b) + (a - b)^2$

5. A map is drawn to a scale of 1 : 40 000.
   (a) Two towns are 18 km apart. Calculate, in cm, their distance apart on the map.
   (b) On the map, a park has an area of 18 cm^2. Calculate, in km^2, the actual area of the park.

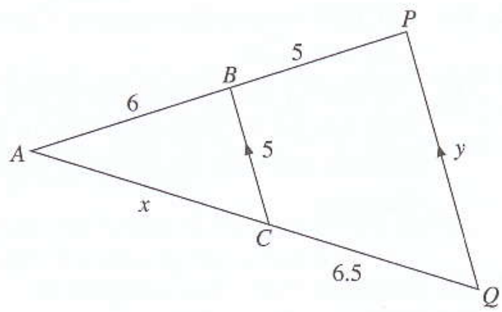
6. Factorise
   (a) $5x^2y - 15xy^2 - 25xy$
   (b) $2ax + 3by - 2ay - 3bx$
   (c) $75x - 27xy^2$
   (d) $9x^2 - 12x + 4$

7. (a) If y is directly proportional to x^2, and if $y = 4$ when $x = 4$, find y when $x = 3$.
   (b) Given that $F = \frac{9}{5}C + 32$, find the value of
      (i) F when $C = 35$,
      (ii) C when $F = 104$.
   (c) Given that y is inversely proportional to x^2 and that $y = 12$ when $x = \frac{1}{2}$, calculate the value of y when $x = 3$.

8. Consider the following number patterns
   $2^3 - 2 = 6 = 1 \times 2 \times 3$
   $3^3 - 3 = 24 = 2 \times 3 \times 4$
   $4^3 - 4 = 60 = 3 \times 4 \times 5$

   (a) Write down the 5th line of the sequence.
   (b) Express $19^3 - 19$ as a product of 3 consecutive numbers.
   (c) Express $x^3 - x$, where x is a whole number, as a product of 3 consecutive numbers.

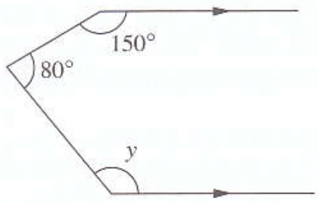
9.

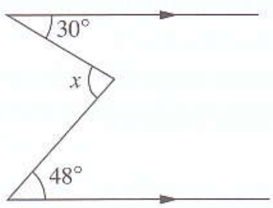


In the figure, CB is parallel to QP. State a pair of similar triangles. Given that $AB = 6$ cm, $BP = BC = 5$ cm, $PQ = y$ cm, $AC = x$ cm and $CQ = 6.5$ cm. Find the value of x and of y.

Revision Exercise I No. 3

1. (a) Find the angles marked x and y in the following figures.





(b) In ΔXYZ, $XY = XZ$ and W is a point on XZ such that $WY = YZ$. If $Y\hat{X}Z = 52°$, find the value of $X\hat{Y}W$.

2. (a) $ABCDEF$ is a regular hexagon. Calculate $B\hat{A}C$ and $A\hat{C}D$.
   (b) Construct a rhombus of sides 5 cm each with one of its diagonals 7.6 cm long. Measure and state the length of the other diagonal.

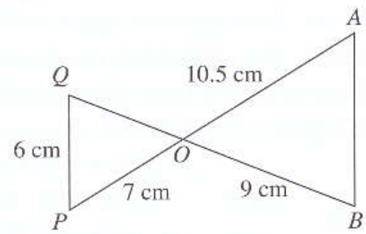
3. (a) Factorise $18x^3 - 8xy^2$ completely.
   (b) Expand and simplify
   $$(x + y)(2x - y) - y^2 + xy$$

4. (a) Factorise $3x^2 - 12$ completely.
   (b) A car travels at a constant speed for 450 km. If the average speed was increased by 15 km/h, the journey would have taken 90 minutes less. Find the speed of the car.
   (c) Two towns X and Y are 460 km apart. A car leaves X for Y at 65 km/h and a slower car leaves Y for X at the same time at 50 km/h. How long will it take the cars to meet each other?

5. (a) Given that y is directly proportional to $(2x - 3)$ and that $y = 21$ when $x = 5$, express y as a function of x. Find the value of
   (i) y when $x = 7$,
   (ii) x when $y = 63$.
   (b) If y is inversely proportional to x^2, and if $y = 2$ when $x = 3$, find the equation relating x and y. Find, also, the value of y when $x = 2$.

6. A map of a town is drawn to a scale of 1 : 20 000.
   (a) A stretch of a highway on the map measures 12.4 cm. Calculate the actual length of this stretch of road in metres.
   (b) A school has an area of 30 000 m². Find the area of this school on the map.

7. In the figure, ΔPOQ is similar to ΔAOB. Given that OP = 7 cm, OA = 10.5 cm, PQ = 6 cm and OB = 9 cm, find the lengths of OQ and AB.



8. (a) Find the number of sides of a regular polygon if each interior angle is 160°.
   (b) If an exterior angle of an octagon is 80° while the other seven exterior angles are each equal to $2x°$, calculate the value of x.

(c) The interior angle of a regular polygon of n sides is 19 times the exterior angle. Find the value of n.

9. (a) Loading half a load of clothes into a washing machine instead of a full load every other day wastes 130 litres of water. Calculate the total amount of water wasted in a year of 366 days if a household washes clothes with a half load. How much extra money will the household have to pay per year, if the cost of water is $2.04 per cubic metre?
   (b) The actual length of a square is 5 cm. A man measures it as 5.2 cm. Find the percentage error in the man's measurement of the area.

10. (a) Find the value of x if
   (i) $x : 4 = 56 : 32$,
   (ii) $9 : 11 = 81 : (100 - x)$.
   (b) Solve the equation
   $\frac{1}{4}(3x - 1) + \frac{1}{6}(7x - 3) = \frac{1}{3}(5x + 2)$.

Revision Exercise I No. 4

1. (a) A man works for 6 days and is paid $153. How much would he be paid for working 9 days at the same rate?
   (b) A cyclist took $3\frac{1}{3}$ hours to cover 46 km. For the first 30 km, he cycled at 15 km/h. Find his speed for the last part of the journey.
   (c) A cyclist travelling at 20 km/h takes 3 hours longer to travel a certain distance than a motorist travelling at 60 km/h. Find the distance travelled.

2. (a) Simplify each of the following:
   (i) $(4a - 3b) - 2(a - b)$
   (ii) $(x + 1)(3x - 5) - (x - 4)(x + 1)$
   (b) Given that $a = 3$, $b = -2$ and $c = 4$, find the value of each of the following:
   (i) $c^2 - 4ab$
   (ii) $ac \div b^2$
   (iii) $3a - b^3 + c$

3. Solve the following equations:
   (a) $\frac{3}{4}(x + 1) + \frac{1}{2}(2x + 1) = 3\frac{1}{4}$
   (b) $2x - 3 = \frac{1}{3}(x - 7)$
   (c) $(2x - 3)^2 = (4x - 1)(x - 6)$

4. (a) Find two consecutive even numbers whose product is 48.
   (b) A father is now four times as old as his son. Five years ago, he was seven times as old as his son. How old are they now?

5. (a) The interior angle of a regular polygon is twice its exterior angle. Find the number of sides of the polygon.
   (b) Four interior angles of a hexagon are 80°, 90°, 100° and 120° while the remaining angles are each equal to x°. Find x.

6. A model of a house is made using a scale of 1 : 100. If the height of the actual gate is 2 m and the area of the hall of the model is 16 cm², calculate
   (a) the height of the gate of the model,
   (b) the area of the actual hall.

7. (a) The length of a rectangle is increased by 20% while its width is decreased by 10%. Calculate the percentage change in the area of the rectangle.
   (b) By watering plants with a container instead of a running hose, Mrs Kumar finds that she can save 115 litres of water per watering session. If Mrs Kumar normally waters plants three times a week, how much water could she save in a year? How much will this water-saving habit translate into money saved

if each litre of water is charged at 0.204 cents? Give your answer correct to the nearest cent. (Assume that there are 52 weeks in a year.)

8. Construct a quadrilateral $PQRS$ where $PQ = PR = 12$ cm, $PS = 10$ cm, $Q\hat{R}P = 64°$ and $R\hat{P}S = 43°$. Measure QR and RS.

9. The scale of a map is 1 : 400 000. Find the area of a piece of land represented by an area of 5.6 cm^2 on the map. Give your answer in km^2.

10. (a) Given that y is directly proportional to x^2 and that when $x = 3$, $y = t$ and when $x = 6$, $y = ct$, find the value of c. Also find the value of y in terms of t when $x = 9$.

(b) Given that y is inversely proportional to the cube of x and that $y = 8t$ when $x = \frac{1}{2}$, find the value of y, in terms of t, when $x = 2\frac{1}{2}$.

Revision Exercise I No. 5

1. (a) A rectangle measures 45 cm by 32 cm. If its width is decreased by 8 cm and its area remains unchanged, find its length.

(b) A cyclist travelling at 14 km/h takes 6 hours 45 minutes to cover a certain journey. How long would it take a car travelling at 52 km/h to cover the same journey? Give your answer correct to the nearest minute.

2. (a) A boy is 42 years younger than his father. In 8 years' time, he will be $\frac{1}{4}$ times as old as his father. Find their present ages.

(b) X can complete a piece of work in 9 days and Y can complete the same work in 18 days. How long will X and Y take to complete the work together?

3. (a) The scale of a map is 1 cm to 500 m.

(i) If the distance between two towns on the map is 8.4 cm, calculate its actual distance in km.

(ii) A railway track has a length of 14.8 km. Calculate its length on the map.

(iii) A town has an area of 4.8 km^2. Find its area on the map.

(b) The R. F. of a map is 1 : 400 000.

(i) Find the distance between two towns on the map which are actually 47 km apart.

(ii) The distance between two towns on the map is 12.8 cm apart. Find the distance on actual ground.

(iii) The area of a housing estate on the map is 3.2 cm^2. Find its actual area, giving your answer in km^2.

4. Expand the following:

(a) $(2x - 1)(3 - 4x)$

(b) $(x + 2)(x^2 - 5)$

(c) $(2x + 3)(2 - 3x - 5x^2)$

5. Factorise the following:

(a) $x^2 - 3x - 10$

(b) $x^2 + 2x - xy - 2y$

* **(c)** $27x^2 - 12y^4$

6. Solve the equations:

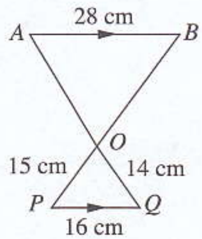
(a) $(x - 3)^2 - 36 = 0$,

(b) $x^2 - 5x - 14 = 0$.

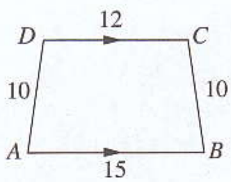
7. (a) Coffee powder of grade A at \$10 per kg and coffee powder of grade B at \$8 per kg are blended in the ratio 3 : 2. Find the cost of 1 kg of the blended coffee.

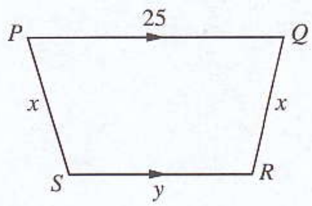
(b) Six of the interior angles of a plane 7-sided polygon are each equal to $x°$ while the remaining angle is $(x + 18)°$. Calculate x.

8. **(a)** In the figure, ΔOBA is similar to ΔOPQ. If $AB = 28$ cm, $PQ = 16$ cm, $OP = 15$ cm and $OQ = 14$ cm, calculate the length of AO and of BO.



(b) In the diagram, trapezium $ABCD$ is similar to trapezium $PQRS$. Calculate the values of x and y.





(c) Construct $\triangle ABC$ in which $AB = 8$ cm, $B\hat{A}C = 46°$ and $A\hat{B}C = 68°$. Measure the length of AC.

9. Expand and simplify

(a) $(2x + 3)(x - 5) - 3x(x - 7)$,

(b) $(3x - 2)(x^2 - 7x + 6) - x(2x^2 - 7)$.

10. Given that y is directly proportional to the positive square root of x and that the value of $y = 10$ when $x = 16$, find

(a) the value of y when $x = 144$,

(b) the value of x when $y = 20$.

In this chapter, you will learn how to

- *manipulate algebraic fractions.*
- *change the subject of a formula.*
- *find the value of an unknown quality in a given formula.*

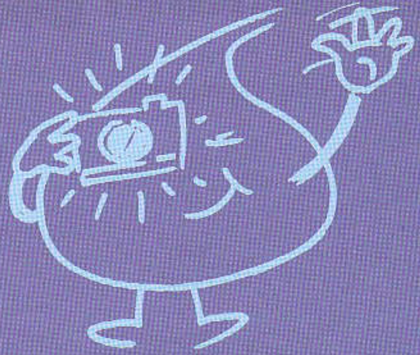
CHAPTER

4

Algebraic Manipulation and Formulae

Introduction

To develop the cameras shown in the picture, engineers have to rely on the important lens formula: $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$. With the knowledge of this formula, they can produce the cameras they want.



Simple Algebraic Fractions

The rules which are associated with numerical fractions are also applicable to algebraic fractions. One important rule to remember is:

> **The value of a fraction remains unchanged if both its numerator and denominator are multiplied or divided by the same non-zero number or expression.**
>
> **i.e.,** $$\frac{a}{b} = \frac{a \times c}{b \times c} \quad \text{and} \quad \frac{a}{b} = \frac{a \div c}{b \div c}, \text{ where } c \neq 0.$$

Example 1

Simplify (a) $\frac{42}{63}$; *(b)* $\frac{24ab^3}{42a^2b}$

Solution

(a) $\frac{\cancel{42}^{\cancel{6}\,2}}{\cancel{63}_{\cancel{9}\,3}} = \frac{2}{3}$

(b) $\frac{{}^{4}\cancel{24}\,\cancel{a}\,b^{\cancel{3}\,2}}{{}_{7}\cancel{42}\,a^{\cancel{2}}\,\cancel{b}} = \frac{4b^2}{7a}$

Notes

We obtain the results shown in Example 1 above by cancelling in the usual way, i.e. by dividing the numerator and the denominator by their common factors.

The final answers should be in the simplest forms, that is, the denominator and numerator have no common factors except 1.

Thinking Time

(a) Is $\frac{a}{b} = \frac{a+c}{b+c}$?

(b) Is $\frac{a}{b} = \frac{a-c}{b-c}$?

Exercise 4a

Simplify the following expressions:

1. $\frac{15x^2}{20x^3}$

2. $\frac{4x^3}{7xy^2}$

3. $\frac{16a^2b^2}{48a^3b}$

4. $\frac{23a^3b}{69ab^3c^2}$

5. $\frac{3m^2np^3}{12mn^3p}$

6. $\frac{15ac^3}{75a^2b^3c}$

7. $\frac{5(xy)^2}{15xy^3}$

8. $\frac{8a^2bc^3}{48a^3b^2c}$

***9.** $\frac{9(a-b)}{27(a-b)^2}$

***10.** $\frac{7a^2(a-3b)^2}{21ab(a-3b)^3}$

***11.** $\frac{4x(x+2y)^3}{24x^3(x+2y)}$

***12.** $\frac{8an(b+c)}{96a^2(b+c)^3}$

Example 2

Simplify the following expressions:

(a) $\dfrac{a^2 + 4ab^2}{3ab}$ (b) $\dfrac{3t}{t^2 - 2t}$ (c) $\dfrac{x^2 - 3x}{3x - 9}$

Solution

(a) $\dfrac{a^2 + 4ab^2}{3ab}$

$= \dfrac{a(a + 4b^2)}{3ab}$

$= \dfrac{a + 4b^2}{3b}$

(b) $\dfrac{3t}{t^2 - 2t}$

$= \dfrac{3t}{t(t - 2)}$

$= \dfrac{3}{t - 2}$

(c) $\dfrac{x^2 - 3x}{3x - 9}$

$= \dfrac{x(x - 3)}{3(x - 3)}$

$= \dfrac{x}{3}$

Pay Attention

Cancellation can usually be done after both the numerator and denominator have been completely factorised. NEVER cancel individual terms of the numerator or the denominator.

Thinking Time

Are the following processes correct? Explain.

$\dfrac{{}^1\cancel{4} + \cancel{10}^2}{{}_2\cancel{8} - \cancel{5}_1} = \dfrac{3}{1}$, $\dfrac{3\cancel{a}^2\cancel{b} + {}^3\cancel{6}\cancel{ab}^2}{\cancel{a}^2 + \cancel{2}\cancel{ab}} = 3 + 3b^2$, $\dfrac{2\cancel{m}^2 - \cancel{4}m}{\cancel{m}^2 - \cancel{4}} = 2 - m$

Exercise 4b

Simplify the following, giving your answers in the simplest forms.

1. $\dfrac{a^2 + 2ac}{4a}$

2. $\dfrac{xy + 3y}{4x + 12}$

3. $\dfrac{de}{d^2 - de}$

4. $\dfrac{2c - 4cd}{6cd}$

5. $\dfrac{6e - 3f}{4e - 2f}$

*6. $\dfrac{3c + 4d}{(6c + 8d)^2}$

7. $\dfrac{c^2 + c}{cd + d}$

8. $\dfrac{3a + 6b}{4ac + 8bc}$

9. $\dfrac{(a - c)^2}{a^2 - ac}$

10. $\dfrac{8a + 4b}{bc + 2ac}$

*11. $\dfrac{2xy + 4y^2}{x^2 + 2xy}$

*12. $\dfrac{m^2}{m^2 - mp}$

*13. $\dfrac{a + b}{a^2 - b^2}$

*14. $\dfrac{x^2 - y^2}{(x - y)^2}$

*15. $\dfrac{(2c - 2d)^2}{(3c - 3d)^2}$

*16. $\dfrac{p^2 - p - 6}{p - 3}$

*17. $\dfrac{c^2 + 7c + 10}{c^2 + 5c}$

*18. $\dfrac{3d - 6}{d^2 + d - 6}$

*19. $\dfrac{a^2 - ab - ac + bc}{a^2 + ab - ac - bc}$

*20. $\dfrac{a^2 + am - an - mn}{a^2 + am + an + mn}$

*21. $\dfrac{q^2 - 2q - 15}{q^2 - 3q - 10}$

*22. $\dfrac{m^2 - 9}{m^2 - 7m + 12}$

Just for fun

During a party, the host distributed balloons to his guests in this way: He gave the first guest 2 balloons plus $\frac{1}{9}$ of the remainder; the second guest, 4 balloons plus $\frac{1}{9}$ of the remainder; the third guest, 6 balloons plus $\frac{1}{9}$ of the remainder; the fourth guest, 8 balloons plus $\frac{1}{9}$ of the remainder; and so on, until the last guest got the final remainder. If all the guests got an equal number of balloons, how many guests were there and how many balloons did the host give out altogether?

Multiplication and Division of Algebraic Fractions

We shall now learn how to multiply and divide algebraic fractions.

Example 3

Simplify (a) $\frac{4}{7} \times \frac{21}{32}$; (b) $\frac{ab}{c^2d} \times \frac{4cd}{6a^2b}$

Solution

(a) $\frac{\cancel{4}^{1}}{\cancel{7}_{1}} \times \frac{\cancel{21}^{3}}{\cancel{32}_{8}} = \frac{3}{8}$

(b) $\frac{\cancel{ab}}{c^2\cancel{d}} \times \frac{\cancel{4}^{2}\cancel{cd}}{\cancel{6}_{3}a^2\cancel{b}} = \frac{2}{3ac}$

Example 4

Simplify (a) $\frac{3}{10} \times \frac{35}{54} \div \frac{14}{15}$ (b) $\frac{x}{y} \times \frac{x^2z}{y^2} \div \frac{xz^2}{2y}$

Solution

(a) $\frac{3}{10} \times \frac{35}{54} \div \frac{14}{15}$

$= \frac{\cancel{3}}{_{2}\cancel{10}} \times \frac{\cancel{35}^{5}}{\cancel{54}_{\cancel{18}_{6}}} \times \frac{\cancel{15}^{5}}{\cancel{14}_{2}}$

$= \frac{5}{24}$

(b) $\frac{x}{y} \times \frac{x^2z}{y^2} \div \frac{xz^2}{2y}$

$= \frac{x}{\cancel{y}} \times \frac{x^2\cancel{z}}{y^2} \times \frac{2\cancel{y}}{\cancel{x}\cancel{z}^{2}}$

$= \frac{2x^2}{y^2z}$

Notes

Any two numbers whose product is 1 are called reciprocals of each other. For example, 3 and $\frac{1}{3}$ are a pair of reciprocals because $3 \times \frac{1}{3} = 1$. We say that the reciprocal of 3 is $\frac{1}{3}$, and the reciprocal of $\frac{1}{3}$ is 3. Other examples of pairs of reciprocals are $\frac{3}{4}$ and $\frac{4}{3}$, -5 and $-\frac{1}{5}$, $-\frac{7}{2}$ and $-\frac{2}{7}$. There is no reciprocal of 0. Similarly, we have reciprocal of x is $\frac{1}{x}$, the reciprocal of $\frac{xz^2}{2y}$ is $\frac{2y}{xz^2}$. So, to divide by a fraction is the same as multiplying by its reciprocal.

Exercise 4c

Simplify the following, giving your answers in the simplest forms.

1. $\frac{15a^2}{8ab^3c} \times \frac{4c}{5ab}$

2. $\frac{12ap^2}{8a^2b^3} \times \frac{16a^3b}{6bp}$

3. $\frac{3x^2}{9x^3y} \div \frac{8xy^2}{24x^2y}$

4. $\frac{3m^2p^3}{14m^3p^2} \div \frac{7m^2p^3}{12mp}$

5. $\frac{9m^2n}{15m^3p^2} \times \frac{21m^3}{10np^3}$

6. $\frac{12x^3}{5ab^3} \div \frac{7x}{15a^2b}$

7. $\frac{a^2b}{4c^2d} \times \frac{2c^3d^3}{3ab^3}$

8. $\frac{5a}{9b} \times \frac{ac^2}{2b} \div \frac{c^3}{8b^3}$

9. $\frac{6uv}{9v^3} \div \frac{8u^3v^2}{27u} \times \frac{3v}{16u^2}$

10. $2y \div \frac{4y}{5xy} \times \frac{64xy}{100x^2y^3}$

11. $\frac{3x^2y^3}{8z^2} \div \frac{9y^2}{10a^2z} \times \frac{6y^2z^3}{5x}$

12. $\frac{12a^3b}{3ab^2} \div \frac{4abc}{3ad} \times \frac{14d^2}{7bc}$

Further Examples on Simplification of Algebraic Fractions

Example 5

Simplify (a) $\frac{4a-16}{a+b} \div \frac{8}{5a+5b}$ ✶(b) $\frac{m^2cx+mc}{m^2x^2+2mx+1}$

Solution

(a) $\frac{4a-16}{a+b} \div \frac{8}{5a+5b}$

$= \frac{4(a-4)}{a+b} \times \frac{5(a+b)}{8}$

$= \frac{5(a-4)}{2}$

✶(b) $\frac{m^2cx+mc}{m^2x^2+2mx+1}$

$= \frac{mc(mx+1)}{(mx+1)^2}$

$= \frac{mc}{mx+1}$

Example 6

Simplify (a) $\dfrac{a^2-2ab}{a-b} \times \dfrac{b-a}{3a-a^2}$

(b) $\dfrac{3b-c}{c+b} \div \dfrac{c-3b}{b+c}$

Solution

(a) $\dfrac{a^2-2ab}{a-b} \times \dfrac{b-a}{3a-a^2}$

$= \dfrac{a(a-2b)}{\cancel{a-b}} \times \dfrac{-\cancel{(a-b)}}{a(3-a)}$

$= -\dfrac{a-2b}{3-a}$

$= \dfrac{a-2b}{a-3}$

(b) $\dfrac{3b-c}{c+b} \div \dfrac{c-3b}{b+c}$

$= \dfrac{3b-c}{c+b} \times \dfrac{b+c}{c-3b}$

$= \dfrac{{}^{1}\cancel{3b-c}}{\cancel{c+b}} \times \dfrac{\cancel{b+c}}{-\cancel{(3b-c)}_{1}}$

$= \dfrac{1}{-1}$

$= -1$

Notes

The following relations are useful when simplifying algebraic expressions.

(a) $a-b=-(b-a)$

(b) $b-a=-(a-b)$

Exercise 4d

Simplify the following fractions:

1. $\dfrac{3(a+b)}{a-b} \times \dfrac{2a-2b}{8a+8b}$

2. $\dfrac{8x^3}{6(x+y)} \div \dfrac{2x^2}{3x+3y}$

3. $\dfrac{h-2k}{16} \div \dfrac{4h-8k}{24}$

4. $\dfrac{3x-7}{5x^2} \div \dfrac{9x-21}{27x}$

5. $\dfrac{6x^2y}{8x-16y} \div \dfrac{4xy^2}{12x-24y}$

6. $\dfrac{x-y}{y+x} \div \dfrac{y-x}{x+y}$

*7. $\dfrac{c^2-d^2}{c^2-2cd+d^2} \div \dfrac{1}{cd+d^2}$

*8. $(a^2-4b^2) \div \dfrac{a^2+2ab}{ab}$

*9. $\dfrac{d^2-4}{d^2-3d+2} \div \dfrac{d}{d-1}$

*10. $\dfrac{m^2-m-6}{m^2-9} \times \dfrac{m^2}{m^2+2m}$

*11. $\dfrac{b^2-4bc+4c^2}{b^2-4c^2} \div \dfrac{c}{b-2c}$

*12. $\dfrac{a^2}{a^2-4} \div \dfrac{3a-a^2}{a^2-5a+6}$

*13. $\dfrac{3(b^2-4)}{9p^2} \times \dfrac{6p^3}{4b+8}$

*14. $\dfrac{y^2-4y+4}{2-6y} \times \dfrac{2y+4}{3y^2-12}$

Explorations

Similar to the least common mutiple (LCM) and the highest common factor (HCF) of integers, define the LCM and HCF of algebraic expressions. Then find the LCM and HCF of each of the following.

1. $3a^2b$ and $4ab^2$
2. $4xyz^2$ and $12x^2yz$
3. $2(x+y)$ and $4(x+y)^2$
4. $4(x-y)$ and $9(x^2-y^2)$
5. $2(a+b)$, $4(a-b)$ and $3(a^2-b^2)$
6. $3x^2+15x+18$ and $4x^2-12x-40$

Addition and Subtraction of Algebraic Fractions

Now, let us learn how to do addition and subtraction involving algebraic fractions.

Your Attention

$$-\frac{a+2b}{4} = \frac{-(a+2b)}{4} = \frac{-a-2b}{4}$$

$$-\frac{a+2b}{4} \neq \frac{-a+2b}{4}$$

Example 7

Simplify $\frac{a+2b}{6} - \frac{a+2b}{4}$.

Solution

$$\frac{a+2b}{6} - \frac{a+2b}{4} = \frac{2(a+2b) - 3(a+2b)}{12} \quad \text{(The LCM of 6 and 4 is 12.)}$$

$$= \frac{2a+4b-3a-6b}{12}$$

$$= \frac{-a-2b}{12}$$

Example 8

Simplify (a) $\frac{2}{c} - \frac{3}{2c}$;

(b) $\frac{2a+3c}{3b} + \frac{a-c}{b}$.

Solution

(a) $\frac{2}{c} - \frac{3}{2c} = \frac{4-3}{2c} = \frac{1}{2c}$ (The LCM of c and $2c$ is $2c$.)

(b) $$\frac{2a+3c}{3b} + \frac{a-c}{b} = \frac{2a+3c+3(a-c)}{3b} \quad \text{(The LCM of } 3b \text{ and } b \text{ is } 3b.\text{)}$$

$$= \frac{2a+3c+3a-3c}{3b}$$

$$= \frac{5a}{3b}$$

Example 9

Simplify $\frac{4}{2x-4y} + \frac{3}{3x-6y}$.

Solution

$$\frac{4}{2x-4y} + \frac{3}{3x-6y}$$

$$= \frac{4}{2(x-2y)} + \frac{3}{3(x-2y)}$$

$$= \frac{2}{x-2y} + \frac{1}{x-2y}$$

$$= \frac{3}{x-2y}$$

Example 10

Simplify $\frac{3}{2-\frac{a}{6}}$.

Solution

$$\frac{3}{2-\frac{a}{6}}$$

$$= 3 \div \left(2 - \frac{a}{6}\right)$$

$$= 3 \div \frac{12-a}{6}$$

$$= 3 \times \frac{6}{12-a}$$

$$= \frac{18}{12-a}$$

Just for fun

Within four consecutive years, Mrs Li gave birth to four lovely children. Today, x years later, Mr and Mrs Li find out that the product of their four children's ages is 3024.

How old is each child now, assuming all of them are of different ages?

Exercise 4e

Express the following as fractions with a single denominator:

1. $\frac{x}{2} + \frac{x-1}{4}$

2. $\frac{1}{3x} - \frac{1}{3y}$

3. $\frac{4x^3y^2}{8xy^2} - \frac{x^2}{4}$

4. $\frac{2y+1}{5} - \frac{3y-2}{10} + \frac{y}{2}$

5. $\frac{c-1}{5} - \frac{2c+3}{3}$

6. $\frac{e-4}{5} + 1$

7. $\frac{2(c+d)}{5} - \frac{2(c-d)}{10}$

8. $\frac{3m}{n} - \frac{m+n}{n}$

9. $\frac{a+3x}{2a} + \frac{a-x}{6a} - \frac{2x+a}{3a}$

10. $\frac{5}{2(e-f)} + \frac{4}{3(f-e)}$

11. $\frac{4c}{10c-5d} + \frac{2d}{6c-3d}$

12. $\frac{a+1}{2a-8} - \frac{a+2}{12-3a}$

13. $\frac{1}{a+\frac{1}{2}}$

14. $\frac{\frac{1}{4}c}{c+\frac{1}{3}}$

* 15. $\frac{\frac{1}{x}+\frac{1}{y}}{\frac{2}{x}}$

Just for fun

Most games that involve skill require the usage of mathematics and logical manipulation.
The following is a simple game that requires mental skill and dexterity, so prepare yourself.
This game requires 2 players only.
There are two piles of toothpicks, known simply as pile A and pile B.
The rules of the game are:
(1) Each player is to remove 1 or more or all the toothpicks from either pile A or pile B.
(2) The loser is the player who takes the last toothpick.
Students should pair up and play the game a few times before coming up with a winning strategy.
Students can start off with a table as shown below:

No of toothpicks in piles		First move by P	Winner P or Q
A	B		
1	1	1 from A	P
1	1	1 from B	P
2	1	1 from A	:
1	2	1 from B	:
2	1	2 from B	:
:	:	:	:

With this, you may be able to come up with a winning strategy.

Further Addition and Subtraction of Algebraic Fractions

The following examples involve addition and subtraction of algebraic fractions with linear or quadratic denominators.

Example 11

Simplify (a) $\dfrac{3}{x+5}+\dfrac{1}{x-2}$ (b) $\dfrac{5}{x^2-4}-\dfrac{2}{x-2}$

Solution

(a) $\dfrac{3}{x+5}+\dfrac{1}{x-2}$

$$= \frac{3(x-2)+1(x+5)}{(x+5)(x-2)}$$

$$= \frac{3x-6+x+5}{(x+5)(x-2)}$$

$$= \frac{4x-1}{(x+5)(x-2)}$$

Notes

The LCM of $(x + 5)$ and $(x - 2)$ is $(x + 5)(x - 2)$.

(b) $\dfrac{5}{x^2-4}-\dfrac{2}{x-2}$

$$= \frac{5}{(x+2)(x-2)}-\frac{2}{x-2}$$

$$= \frac{5-2(x+2)}{(x+2)(x-2)}$$

$$= \frac{5-2x-4}{(x+2)(x-2)}$$

$$= \frac{1-2x}{(x+2)(x-2)}$$

Notes

We factorise (x^2-4) before simplification.
The LCM of $(x + 2)(x - 2)$ and $(x - 2)$ is $(x + 2)(x - 2)$.

Exercise 4f

1. Simplify each of the following.

(a) $\frac{1}{a} - \frac{2}{a+b}$

(b) $\frac{5}{a} + \frac{3}{a+2}$

(c) $\frac{2}{x+5} - \frac{3}{x-1}$

(d) $\frac{4}{x-5} + \frac{2}{2x+1}$

(e) $\frac{7}{2x-5} + \frac{4}{x-3}$

(f) $\frac{1}{3x+2} - \frac{5}{2x-7}$

* (g) $\frac{3x}{3x-5} - \frac{4x}{4x-1}$

(h) $\frac{3}{5x+1} - \frac{2}{5x-1}$

* (i) $\frac{3x}{6x+7} - \frac{2}{5x-4}$

(j) $\frac{11}{3x-7} + \frac{2}{6-5x}$

2. Simplify each of the following.

(a) $\frac{3}{x^2-1} + \frac{2}{x-1}$

(b) $\frac{5}{x^2-9} - \frac{7}{x+3}$

(c) $\frac{3}{4x^2-1} - \frac{5}{2x+1}$

* (d) $\frac{3x}{9x^2-4} + \frac{5}{3x+2}$

* (e) $\frac{3}{4x-5} - \frac{7x}{16x^2-25}$

(f) $\frac{2}{x-2} + \frac{3}{(x-2)^2}$

* (g) $\frac{x+5}{x^2-6x} - \frac{3}{x-6}$

* (h) $\frac{5}{2x+1} - \frac{2x}{(2x+1)^2}$

(i) $\frac{3}{x^2-3x} - \frac{1}{x-3}$

* (j) $\frac{4}{3x-5} - \frac{2x}{(3x-5)^2}$

3. Express the following as a fraction with a single denominator.

(a) $\frac{a+b}{a-b} + \frac{a^2-4b^2}{a^2-b^2} - \frac{a-3b}{a+b}$

(b) $\frac{5a}{a-9} + \frac{a^2-2a+1}{a^2-12a+27} - \frac{6a}{a-3}$

(c) $\frac{8x^2+18y^2}{4x^2-9y^2} - \frac{2x+3y}{2x-3y} + \frac{2x-3y}{2x+3y}$

* (d) $\frac{x}{x^2+xy} - \frac{y}{x^2-y^2} + \frac{x+y}{xy-y^2}$

* (e) $\frac{2}{m-4} + \frac{1}{m} + \frac{3}{m-3}$

(f) $\frac{m-4}{2m-1} + \frac{5m^2+9m+14}{2m^2+3m-2} - \frac{3m-5}{m+2}$

* (g) $\frac{2}{a-1} + \frac{1}{(a-1)^2} + \frac{2}{(a-1)^3}$

* (h) $\frac{4}{a-1} - \frac{3}{a-2} - \frac{1}{a-3}$

(i) $\frac{2}{a^2-1} - \frac{1}{a+1} - \frac{1}{a-1}$

(j) $\frac{1}{2a-3} - \frac{2}{3-2a} + \frac{18}{9-4a^2}$

Equations Involving Algebraic Fractions

We shall now learn how to solve equations involving algebraic fractions.

Example 12

Solve the equation $\frac{b-2}{5} + \frac{b-1}{3} = 1$.

Solution

$\frac{b-2}{5} + \frac{b-1}{3} = 1$

Multiply by the LCM of 5 and 3 throughout, i.e., multiply by 15:

$$\begin{aligned} 3(b-2) + 5(b-1) &= 15 \\ 3b - 6 + 5b - 5 &= 15 \\ 8b &= 15 + 5 + 6 \\ 8b &= 26 \\ b &= \frac{26}{8} = 3\frac{1}{4} \end{aligned}$$

Example 13

Solve $\frac{6}{2b-5} - \frac{4}{b-3} = 0$.

Solution

$\frac{6}{2b-5} - \frac{4}{b-3} = 0$

Multiply by $(2b-5)(b-3)$ throughout:

$$\begin{aligned} 6(b-3) - 4(2b-5) &= 0 \\ 6b - 18 - 8b + 20 &= 0 \\ -2b + 2 &= 0 \\ b &= 1 \end{aligned}$$

Check:
We need $2b - 5 \neq 0$ and $b - 3 \neq 0$ which means $b \neq 2\frac{1}{2}$ and $b \neq 3$.

Exercise 4g

1. Solve the following equations:

(a) $\frac{2x-3}{5} = \frac{x-2}{2}$

(b) $\frac{m}{m+2} = \frac{3}{5}$

(c) $\frac{2}{x-3} = 4$

(d) $\frac{x}{2} + \frac{x-2}{3} = 4$

(e) $\frac{7p}{8} + 3 = \frac{9p}{10}$

(f) $1 + \frac{5}{e} = \frac{7}{3}$

(g) $e + \frac{e}{2} + \frac{e}{3} = 11$

(h) $\frac{5}{a+4} - \frac{2}{a-2} = 0$

(i) $\frac{x+3}{4} = \frac{2x-3}{5}$

(j) $6 + \frac{2x-1}{4} = x$

2. Solve the following equations:

(a) $\frac{3m-1}{5} - \frac{2m-3}{3} = 1$

(b) $\frac{x}{4} - \frac{5x+8}{6} = \frac{2x-9}{3}$

(c) $\frac{m}{2} + \frac{m+1}{7} = m - 2$

(d) $\frac{2a-3}{2} - \frac{a+2}{3} = \frac{a+1}{4}$

(e) $\frac{x+2}{6} + \frac{x}{3} = \frac{2-x}{2}$

(f) $\frac{4x+1}{5} - \frac{x}{2} = \frac{x}{3} - \frac{x}{4}$

(g) $\frac{2a+3}{2} + \frac{a}{4} = \frac{a}{6} - \frac{2a}{3}$

(h) $\frac{d+3}{3} - \frac{2d-3}{2} = d - \frac{5}{6}$

(i) $\frac{3}{x+1} - \frac{1}{2x+2} = 5$

(j) $\frac{5}{6x} + \frac{6}{7x} - \frac{9}{14x} = 4$

Problem Solving Involving Algebraic Fractions

Problems involving algebraic fractions may be solved using the various problem solving heuristics we learnt earlier.

Let's look at the following problem.

Example 14

Mrs Li bought some oranges. She gave $\frac{1}{2}$ of them to her sister, $\frac{1}{4}$ of the remainder to her neighbour, $\frac{3}{5}$ of those left to her children and had 6 left in the end. How many oranges did Mrs Li buy?

Solution

One method that you can use is by working backwards.

Strategy 1: Work backwards

(1) The last 6 oranges represent $\frac{2}{5}$ of those left since Mrs Li gave $\frac{3}{5}$ of those left to her children.

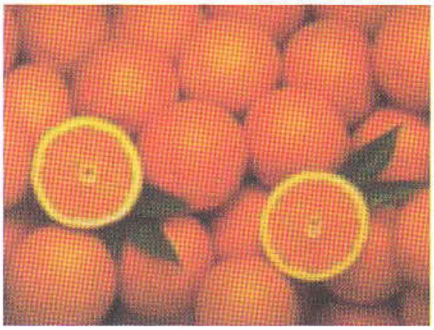
$\therefore$ She had $6 \times \frac{5}{2} = 15$ oranges left after giving some to her neighbour.

(2) Her neighbour received $\frac{1}{4}$ of the remaining oranges after Mrs Li gave $\frac{1}{2}$ of the oranges she bought to her sister. Thus the 15 oranges represent the other $\frac{3}{4}$.

$\therefore$ Mrs Li had $15 \times \frac{4}{3} = 20$ oranges left after giving half to her sister.

(3) Since her sister received $\frac{1}{2}$ of the oranges bought, this means that the 20 oranges represent the other half.

$\therefore$ Mrs Li bought $20 \times 2 = 40$ oranges.



Just For Fun

Teri says to David, "I can tell you when your birthday is if you tell me the answer to the following:

1. *Multiply the number of your birth month by 5. (If you were born in September, then multiply 9 by 5, etc.)*
2. *Add 7.*
3. *Multiply the result by 4.*
4. *Add 13.*
5. *Multiply that total by 5.*
6. *Add the number of the day on which your birthday falls. (If you were born on 3rd April, add 3, etc.)*
7. *Subtract 205 from the total.*

Now, your answer will be either a three- or a four-digit number. If it is a three-digit number, the first digit will be the month and the next two digits will be the day of your birthday. If it is a four-digit number, the first two digits will be the month and the last two digits will be the day of your birthday. Am I right?"

How does this trick work?

This problem can also be solved by the algebraic method.

Strategy 2: Use the algebraic method

Step 1: Analysing the question

Let x be the number of oranges bought.

x

- to her sister: $\frac{1}{2}x$
- remainder: $\frac{1}{2}x$
  - to her neighbour: $\frac{1}{4} \times \frac{1}{2}x = \frac{1}{8}x$
  - remainder: $\frac{3}{4} \times \frac{1}{2}x = \frac{3}{8}x$
    - children: $\frac{3}{5} \times \frac{3}{8}x = \frac{9}{40}x$
    - remainder: $\frac{2}{5} \times \frac{3}{8}x = \frac{3}{20}x$

Step 2: Forming an equation

$$\therefore \quad \frac{3}{20}x = 6$$

Step 3: Solving the equation

$$x = 6 \times \frac{20}{3} = 40$$

Step 4: Check the solution

$40 \times \frac{1}{2} = 20 \longrightarrow$ to her sister,

$20 \times \frac{1}{4} = 20 \longrightarrow$ to her neighbour,

$15 \times \frac{3}{5} = 9 \longrightarrow$ to her children,

$15 - 9 = 6 \longrightarrow$ remainder

The solution is correct.

Generally, the algebraic method consists of 4 steps:

Step 1: Let the unknown be denoted by a variable.
Step 2: Form an equation involving the variable.
Step 3: Solve the equation.
Step 4: Check the solution.

Example 15

When a number is subtracted from 52 and the result is divided by 6, the answer obtained is twice the original number. What is the number?

Solution

Let the number be a.

$$\frac{52 - a}{6} = 2a$$
$$52 - a = 12a$$
$$52 = 13a$$
$$\frac{52}{13} = a$$
$$a = 4$$

Check:
$52 - 4 = 48$ and $48 \div 6 = 8$;
8 is twice of 4.

Example 16

A motorist took $3\frac{1}{3}$ hours to drive 160 km. He drove part of the way at an average speed of 50 km/h and the rest of the way at 45 km/h. What is the distance he travelled at 50 km/h?

Solution

Let x km be the distance he travelled at 50 km/h.

Hence, time taken to travel x km = $\frac{x}{50}$ h.

He travelled $(160 - x)$ km at 45 km/h.

Time taken to travel $(160 - x)$ km = $\frac{160 - x}{45}$ h.

The total time taken was $3\frac{1}{3}$ hours.

$$\frac{x}{50} + \frac{160 - x}{45} = 3\frac{1}{3}$$
$$\left(\frac{x}{50} \times 450\right) + \left(\frac{160 - x}{45} \times 450\right) = \left(\frac{10}{3} \times 450\right)$$
$$9x + 10(160 - x) = 1500$$
$$9x + 1600 - 10x = 1500$$
$$-x = 1500 - 1600$$
$$x = 100$$

Check:
100 km at 50 km/h requires 2 hours.

Thus $(160 - 100) = 60$ km at 45 km/h requires $1\frac{1}{3}$ hours.

$2\text{ h} + 1\frac{1}{3}\text{ h} = 3\frac{1}{3}\text{ h}$

Example 17

Mr Tan makes monthly visits to his parents in Malacca, a distance of 240 km from Singapore. He finds that if he increases the average speed by 10 km/h, he could save a total of 20 minutes for the journey. Find the original speed of Mr Tan.

Solution

Let the speed at which Mr Tan normally travels be x km/h.

Time taken by Mr Tan = $\frac{240}{x}$ hours.

When the speed = $(x + 10)$ km/h, the time taken = $\frac{240}{x+10}$ hours.

We have $\frac{240}{x} - \frac{240}{x+10} = \frac{20}{60}$

i.e. $\frac{240(x+10) - 240x}{x(x+10)} = \frac{1}{3}$

$$x(x + 10) = 3(2\,400)$$
$$x^2 + 10x - 7\,200 = 0$$
$$(x - 80)(x + 90) = 0$$
$$\therefore \quad x = 80 \quad \text{or} \quad x = -90 \quad \text{(not applicable)}$$

Thus, the original speed of Mr Tan is 80 km/h.

Exercise 4h

1. When 25 is added to a number and the result is halved, the answer is 3 times the original number. What is the number?

2. When 18 is added to $\frac{1}{4}$ of a number, the result is $2\frac{1}{2}$ times the original number. What is the number?

3. A number, when added to 5, gives the same result as when $\frac{2}{3}$ of it is subtracted from 6. What is the number?

4. 5 is subtracted from 4 times a number and the result is then doubled. If the answer is 6, what is the original number?

5. When a number is added to 4, the result is equal to subtracting 10 from three times of it. What is the number?

6. How can the number 39 be divided into two parts in order that the sum of $\frac{2}{3}$ of one part and $\frac{3}{4}$ of the other part is 28?

7. Meiling and Meimei divide \$69 into 2 shares. $\frac{3}{4}$ of Meiling's share is equal to $\frac{2}{5}$ of Meimei's share. How much does each get?

8. A mother is 21 years older than her new born daughter. How old will the daughter be when her age is $\frac{1}{4}$ that of her mother's?

9. Mary's age is $\frac{2}{3}$ that of Peter's. Two years ago Mary's age was $\frac{1}{2}$ of what Peter's age will be in 5 years' time. How old is Peter now?

10. A half of what John's age was 4 years ago is equal to one third of what it will be in 5 years' time. How old is John now?

11. If a piece of wood is 5 cm longer than a second piece, and $\frac{3}{4}$ of the second piece is equal to $\frac{3}{5}$ of the first, what is the length of the second piece?

12. A man walked for some distance at 8 km/h, and for an equal distance at 5 km/h. The total time he took was $3\frac{1}{4}$ hours. Find the total distance he walked.

13. A man cycled for some time at 16 km/h and returned at 15 km/h. The total time taken was $7\frac{3}{4}$ hours. Find the total distance he cycled.

14. A cyclist goes from one village to another at 28 km/h. He returns at 24 km/h. If the return journey takes two hours longer than the outward journey, what is the distance between the villages?

15. A boy has a certain number of sweets. If he eats 16 a day, they will last him 2 days longer than if he eats 18 a day. How many sweets has he?

16. A certain number of matches are needed to fill 24 boxes, with each box containing the same number of matches. When 4 less matches are put into each box, there are enough for 28 boxes. Find the total number of matches.

17. Mr Ong makes regular business trips to Kuala Lumpur, a distance of 420 km from Singapore.

(a) On his way up to Kuala Lumpur he travels along the trunk road at an average speed of x km/h. Write down an expression, in terms of x, for the time taken, in hours, to travel from Singapore to Kuala Lumpur.

(b) On his return journey to Singapore, he travels along the North South Highway and increases the average speed by 15 km/h. Write down an expression, in terms of x, for the time taken, in hours, to travel from Kuala Lumpur to Singapore.

(c) If the time difference between the two journeys is 40 minutes, form an equation in x and show that it reduces to $x^2 + 15x - 9450 = 0$.

(d) Solve the equation and find the time taken for the trip from Singapore to Kuala Lumpur.

18. Mr Kumar lives in the eastern part of Singapore. He visits his aged parents, who lives 36 km away, every weekend. He finds that if he increases the average speed of his vehicle by 12 km/h, he could save 9 minutes of his travelling time. Find the speed at which he travels before the increase in speed.

19. A man sold x similar books for \$132, If he had sold $(x - 1)$ books but charges \$1 more for each book, he would have received the same amount of money. Find the value of x.

20. Adam works part time at a fast food restaurant that pays \$$x$ per hour, while Jenny works as a sales assistant in a boutique that pays \$$(x + 2)$ per hour. Adam works 8 hours more per week than Jenny and they each earn \$192 a week. Find the value of x and the number of hours that Adam works in a week.

Changing the Subject of a Formula

Consider the following two sentences:

(1) Michael is the younger brother of Simon.
(2) Simon is the elder brother of Michael.

These two sentences actually have the same meaning.

The only difference is that 'Michael' is the subject of the first sentence and is put at the beginning of the sentence while 'Simon' is the subject of the second sentence and is placed at the beginning of the second sentence.

Similarly, an algebraic formula may be expressed differently to suit a particular purpose. For example, the formula for the area of a rectangle is $A = lb$ where A is the area of the rectangle, l its length and b its breadth. We say that A is the subject of the formula.

If we need to find the length of the rectangle, we can rearrange the formula to make l the subject of the formula.

$A = lb$

$\therefore \quad l = \dfrac{A}{b}$ Divide both sides by b.

Similarly, to find the breadth, we can make b the subject of the formula.

$b = \dfrac{A}{l}$ Divide both sides by l.

The three formulae $A = lb$, $l = \dfrac{A}{b}$ and $b = \dfrac{A}{l}$ are equivalent. They may be used for different purposes.

The following examples illustrate the technique of changing the subject of a formula.

Just For Fun

1. If ABCDE × 4 = EDCBA, find A, B, C, D and E where none of them is zero.

$$\begin{array}{r} ABCDE \\ \times \quad 4 \\ \hline EDCBA \end{array}$$

2. If $\dfrac{PORK}{CHOP} = C$ and $C > 2$, find the value of each of these letters.

For Your Information

Algebra is a branch of mathematics which deals with relations and properties of numbers by means of letters and other general symbols.

Example 18

Make P the subject of the formula $I = \dfrac{PRT}{100}$.

Solution

$$I = \frac{PRT}{100}$$

$PRT = 100I$ (Multiply both sides by 100.)

$P = \dfrac{100I}{RT}$ (Divide both sides by RT.)

Example 19

Make p the subject of the formula $3b = 2p - 7$.

Solution

$$3b = 2p - 7$$

$2p = 3b + 7$ (Add 7 to both sides.)

$p = \dfrac{3b + 7}{2}$ (Divide both sides by 2.)

Example 20

Make x the subject of the formula $y = \dfrac{2 - x}{3 + 2x}$.

Solution

$$y = \frac{2 - x}{3 + 2x}$$

$y(3 + 2x) = 2 - x$ (Multiply both sides by $(3 + 2x)$.)

$3y + 2xy = 2 - x$

$2xy + x = 2 - 3y$

$x(2y + 1) = 2 - 3y$ (Factorise the left-hand side.)

$x = \dfrac{2 - 3y}{2y + 1}$ (Divide both sides by $(2y + 1)$.)

Just for fun

Each of the letters below represents a certain value. Try figuring out the value of each letter and then work out the sums.

(a)
```
   S E N D
+  M O R E
 M O N E Y
```

(b)
```
   W I R E
+  M O R E
 M O N E Y
```

There is only one answer for (a) but more than one possible answer for (b).

Exercise 4i

1. Make a the subject of the given formulae:

   (a) $ax = y$

   (b) $a(p - 4) = q$

   (c) $ax + by = c$

   (d) $p(a + b) = c$

   (e) $2a - 3m = 4a - 7$

   (f) $5b - 2a = 3c$

   (g) $\frac{a}{m} + b = c$

   (h) $x = \frac{2a}{3} + 5z$

   (i) $\frac{p + a}{5} = 3p$

   (j) $R = m(a + g)$

2. Make the letter in the brackets the subject of the given formulae:

   (a) $A = \frac{1}{2}bh$ (b)

   (b) $T = \frac{kx^2}{2l}$ (k)

   (c) $\frac{F}{m} = \frac{v - u}{t}$ (t)

   (d) $3k = \frac{12 + 2l}{5}$ (l)

   (e) $\frac{a}{b} - \frac{a}{c} = 1$ (a)

   (f) $\frac{a}{3} + \frac{b}{4} = \frac{c}{5}$ (a)

   (g) $A = \frac{h}{2}(a + b)$ (a)

   (h) $S = \frac{n}{2}(a + l)$ (l)

   (i) $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$ (q)

   (j) $s = ut + \frac{1}{2}gt^2$ (u)

3. Make the letter in the brackets the subject of the formulae:

   (a) $p = 2(l + b)$ (l)

   (b) $A = 2\pi r(r + h)$ (h)

   (c) $a(c - 3) = bc$ (c)

   (d) $a + b = \frac{ax}{c}$ (a)

   (e) $\frac{1}{a} + \frac{1}{b} = 1$ (b)

   (f) $a = \frac{5}{4}(b - 18)$ (b)

   (g) $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$ (b)

   (h) $\frac{a}{2c} + \frac{b}{4} = 1$ (a)

   (i) $\frac{1}{a} + \frac{1}{b} = \frac{b}{a}$ (a)

   (j) $x = \frac{y(z - y)}{z}$ (z)

   (k) $\frac{b - x}{a} = \frac{x}{c}$ (x)

   (l) $x = \frac{1}{y + 1} + 2$ (y)

Further Examples on Changing the Subject of a Formula

We shall now learn how to change the subject of a formula involving squares, cubes and their roots.

Example 21

(a) If $\sqrt{a + l} = 2b$, make a the subject of the formula.

(b) If $c + 7 = \frac{x^2}{3}$, make x the subject of the formula.

Solution

(a) $\sqrt{a + l} = 2b$

$a + l = (2b)^2$

$\therefore \quad a = 4b^2 - l$

(b) $c + 7 = \frac{x^2}{3}$

$x^2 = 3(c + 7)$

$x^2 = 3c + 21$

$\therefore \quad x = \pm\sqrt{3c + 21}$

Notes

When solving problems that involve square roots such as that illustrated in Example 21(a), square both sides of the equations.

For example 21 (b), multiply both sides of the equation by 3 (to get rid of the denominator) and then take square roots on both sides of the equation to obtain a value of x.

Example 22

(a) Make s the subject of the formula $v = \sqrt{u^2 + 2as}$.

(b) Make x the subject of the formula $\sqrt[3]{ax + b} = k$.

Solution

(a) $v = \sqrt{u^2 + 2as}$

$v^2 = u^2 + 2as$

$2as = v^2 - u^2$

$\therefore \quad s = \frac{v^2 - u^2}{2a}$

(b) $\sqrt[3]{ax + b} = k$

$ax + b = k^3$

$ax = k^3 - b$

$\therefore \quad x = \frac{k^3 - b}{a}$

Notes

For Example 22(a), square both sides of the equation. The value of s can be derived when both sides of the equation are divided by 2a.

For Example 22(b), cube both sides of the equation.

Exercise 4j

1. Make a the subject of the formula for each of the following:

(a) $\sqrt{a} = b$

(b) $\sqrt{2a} = b$

(c) $e = \sqrt{5a-8}$

(d) $\sqrt{\frac{a}{2}} = b$

(e) $x = \sqrt{\frac{2a}{5c}}$

(f) $\sqrt{3a-2} = \sqrt{\frac{a}{b}}$

(g) $2a^2 = b - 3$

(h) $A = 4\pi a^2$

(i) $\sqrt[3]{a-b} = c$

(j) $K = \frac{0.5\ ma^2}{nb}$

2. Make the letter in the brackets the subject of the given formula:

(a) $a = \sqrt{a+2b}$ $\quad(b)$

(b) $(x+y)^2 = x$ $\quad(y)$

(c) $x = 2w^2 + b$ $\quad(w)$

(d) $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$ $\quad(b)$

(e) $t = \sqrt{\frac{4x}{m-3}}$ $\quad(x)$

(f) $\frac{1}{a} - \frac{1}{b} = \frac{1}{c-2}$ $\quad(c)$

(g) $t = \frac{3x - 8mg}{12kg}$ $\quad(g)$

(h) $D = \sqrt{b^2 - 4ac}$ $\quad(b)$

(h) $2ps^2 = 3qs^2 + 5p$ $\quad(s)$

(j) $\frac{p}{q} = \frac{1}{3n}\sqrt{\frac{h+2k}{3h+k}}$ $\quad(h)$

3. Make the letter in the brackets the subject of the formulae:

(a) $p + \frac{p}{q} = q$ $\quad(p)$

(b) $l^2 = \frac{k}{m} - \frac{a}{2m^2}$ $\quad(a)$

(c) $w = \frac{ar}{R - r}$ $\quad(r)$

(d) $y = \frac{3 - x}{1 + 2x}$ $\quad(x)$

(e) $b = \frac{abc}{4 - c}$ $\quad(c)$

(f) $\frac{m}{p - q} = r$ $\quad(q)$

(g) $p - q = \frac{2p}{3}$ $\quad(p)$

(h) $\frac{b}{2} = \frac{5 - x}{2 + 3x}$ $\quad(x)$

Finding an Unknown in a Formula

We have learnt that the formula for the volume of a cube is $V = l^3$, where l is the length of the cube.

We can find the volume of the cube, V, if we are given the value of the length, l. If $l = 2$, $V = 2^3 = 8$. If $l = 4$, $V = 4^3 = 64$.

We can also find the value of l when we are given the value of V. If $V = 1000$, $1000 = l^3$. Taking the cube root of both sides, we have $l = 10$.

Just for fun

1. Find three distinct positive integers x, y and z such that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.$$

2. Make each of the following equations correct by filling in +, −, ×, ÷, ( ) or { } in the appropriate places.

1. 1 2 3 = 1
2. 1 2 3 4 = 1
3. 1 2 3 4 5 = 1
4. 1 2 3 4 5 6 = 1
5. 1 2 3 4 5 6 7 = 1
6. 1 2 3 4 5 6 7 8 = 1
7. 1 2 3 4 5 6 7 8 9 = 1

Example 23

Given that $y = \sqrt{\frac{64}{3x+1}}$, *find the value of*

(a) y when x = 1,
(b) x when y = 2.

Solution

(a) $y = \sqrt{\frac{64}{3x+1}}$

When $x = 1$, $y = \sqrt{\frac{64}{3(1)+1}} = 4$

(b) When $y = 2$, $2 = \sqrt{\frac{64}{3x+1}}$

Squaring both sides, $4 = \frac{64}{3x+1}$

$\therefore\ 3x + 1 = \frac{64}{4} = 16$

$x = 5$

Just for fun

Find the possible values of X, Y and Z if XY × XY = ZZY.

Example 24

(a) *Given that* $x=\sqrt{a^2+b^2}$, *find the values of a when x = 17 and b = 8.*

(b) *Given that* $\sqrt[3]{\frac{x+y}{x-y}}=z$, *find the value of x when y = 4 and z = 3.*

Solution

(a) $x=\sqrt{a^2+b^2}$

$17=\sqrt{a^2+8^2}$

Squaring both sides, $17^2 = a^2+64$

$a^2=17^2-64=225$

$\therefore\ a=\pm\sqrt{225}=\pm 15$

Notes

If a formula involves more than two variables, the value of an unknown can only be found when the values of the other variables are given.

(b) $\sqrt[3]{\frac{x+y}{x-y}}=z$

$\sqrt[3]{\frac{x+4}{x-4}}=3$

Cubing both sides, $\frac{x+4}{x-4}=3^3=27$

$27(x-4)=x+4$ (Multiply both sides by $(x-4)$)

$27x-108=x+4$

$26x=112$

$x=\frac{112}{26}=4\frac{4}{13}$

Exercise 4k

1. Given that $x+\sqrt{y+z}=k$, find the value of y when $x=3$, $z=4$ and $k=10$.

2. Given that $\sqrt{ax^2-b}=c$, find the value(s) of x when $a=2$, $b=7$ and $c=5$.

3. Given that $a\sqrt{b^2-c}=5k$, find the value of
   (a) k when $a=3$, $b=6$ and $c=20$,
   (b) c when $a=4$, $b=7$ and $k=11$.

4. Given that $a=\sqrt{\dfrac{3b+c}{b-c}}$, find the value(s) of
   (a) a when $b=7$ and $c=2$,
   (b) c when $b=9$ and $a=4$.

5. Given that $a^2+a^2b=320$, find the value(s) of
   (a) b when $a=8$,
   (b) a when $b=2\frac{1}{5}$.

6. Given that $\dfrac{m(nx-y^2)}{p}=3n$, find the value (s) of
   (a) p when $m=5$, $n=7$, $x=4$ and $y=-2$,
   (b) n when $p=9$, $m=14$, $x=2$ and $y=3$,
   (c) y when $p=15$, $n=4$, $x=42$ and $m=5$.

7. Given that $A=\frac{1}{3}\pi r^2h+\frac{4}{3}\pi r^3$, find the value of
   (a) A when $r=7$, $h=15$ and $\pi=3.142$,
   (b h when $A=15400$, $r=14$ and $\pi=3.142$.

8. Given that $V=\pi R^2h+\frac{2}{3}\pi r^3$, find the value of
   (a) V when $\pi=3.142$, $R=12$, $r=9$ and $h=14$,
   (b) h when $\pi=3.142$, $R=8$, $r=6$ and $V=3800$,
   (c) R when $\pi=3.142$, $h=17$, $r=8.5$ and $V=3500$,
   (d) r when $\pi=3.142$, $R=11$, $h=6.9$ and $V=4600$.

9. Given that $y=3x+\sqrt[3]{a+b^2}$, find the value(s) of
   (a) y when $x=3.8$, $a=13$ and $b=15$,
   (b) a when $x=8.5$, $b=13$ and $y=35$,
   (c) b when $y=56$, $x=15.6$ and $a=23$.

SUMMARY

1. The value of a fraction remains unchanged if both its numerator and its denominator are multiplied or divided by the same non-zero number or expression i.e. $\frac{a}{b} = \frac{a \times c}{b \times c}$ and $\frac{a}{b} = \frac{a \div c}{b \div c}$.

2. Generally, the algebraic method for solving a problem consists of the following steps:
   (a) Let the unknown be denoted by a variable.
   (b) Form an equation involving the variable.
   (c) Solve the equation.
   (d) Check the solution.

Review Examples 4

Example 1

A man travels regularly between two towns. If he travels at his usual average speed, he would take $4\frac{2}{3}$ *hours. He finds that if he increases his average speed for the journey by 3 km/h, he can reduce the time taken by 20 minutes. What is his normal average speed?*

Solution

Let the man's usual average speed be x km/h

$\therefore$ the distance between the two towns = $4\frac{2}{3} \times x$ km

His new average speed will now be $(x + 3)$ km/h and the new time taken

$= \left(4\frac{2}{3} - \frac{20}{60}\right)$ h $= 4\frac{1}{3}$ h.

The distance between the towns = $4\frac{1}{3}(x + 3)$ km

$$4\frac{2}{3}x = 4\frac{1}{3}(x + 3)$$

$$4\frac{2}{3}x = 4\frac{1}{3} + 13)$$

$$\text{i.e. } \frac{1}{3}x = 13$$

i.e. the man's usual speed is 39 km/h.

Example 2

Given that $E = mgh + \frac{1}{2}mv^2$, *express v in terms of E, m, g and h. Hence find the value(s) of v when m = 6, g = 10, h = 30 and E = 3000.*

Solution

$$E = mgh + \frac{1}{2}mv^2$$

$$\frac{1}{2}mv^2 = E - mgh$$

$$v^2 = \frac{2(E - mgh)}{m}$$

$$v = \pm\sqrt{\frac{2(E - mgh)}{m}}$$

When $m = 6$, $g = 10$, $h = 30$ and $E = 3000$,

$$v = \pm\sqrt{\frac{2(3000 - 6 \times 10 \times 30)}{6}}$$

$$= \pm 20$$

Example 3

A shop proprietor bought x CD-Roms for \$108.

(a) Write down an expression in terms of x, for the cost price, in dollars, of one CD-Rom.

(b) The proprietor priced each CD-Rom to sell at a profit of \$1.50. Find an expression, in terms of x, for the selling price of each CD-Rom.

(c) Given that he was able to sell only 10 CD-Roms at this price and the remaining CD-Roms were sold at a price of \$7.50 each, find an expression, in terms of x, for the total amount of money he received for the CD-Roms.

(d) If the proprietor received \$120 altogether, form an equation in x and show that it reduces to $x^2 - 24x + 144 = 0$. *Hence find the value of x.*

Solution

(a) The price of each CD-Rom = $\$\frac{108}{x}$.

(b) The selling price of each CD-Rom $= \$\left(\frac{108}{x} + \frac{3}{2}\right)$

$$= \$\frac{216 + 3x}{2x}$$

(c) The total amount he received for the CD-Roms

$$= \$10\left(\frac{216 + 3x}{2x}\right) + (x - 10)7.50$$

$$= \$\frac{1080 + 15x + 7.5x^2 - 75x}{x}$$

(d) We have $\frac{1080 + 15x + 7.5x^2 - 75x}{x} = 120$

$$\begin{aligned} \text{i.e.} \quad 1\,080 + 7.5x^2 - 60x &= 120x \\ 7.5x^2 - 180x + 1\,080 &= 0 \\ x^2 - 24x + 144 &= 0 \\ (x - 12)(x - 12) &= 0 \\ \therefore \quad x &= 12. \end{aligned}$$

Review Questions 4

1. Express each of the following as a single fraction in its simplest form.

(a) $\frac{1}{x} + \frac{1}{3x}$

(b) $\frac{x-2}{3} - \frac{2x-3}{9}$

(c) $\frac{x+3}{5} - \frac{1-x}{6}$

(d) $\frac{2}{x} + \frac{3}{4x}$

(e) $\frac{10}{3x} - \frac{4}{x}$

(f) $\frac{x}{2x-1} - \frac{4}{5}$

(g) $3x + \frac{2}{x}$

(h) $\frac{9x+8}{5} + 2$

(i) $\frac{2}{2x-1} - \frac{3}{5x+1}$

(j) $\frac{3}{x+1} - \frac{2}{3x+4}$

(k) $\frac{1}{x-2} + \frac{5}{3x-7}$

(l) $\frac{2}{x+y} - \frac{3}{x-y}$

(m) $\frac{5}{2x-1} + \frac{3}{2x+1}$

(n) $\frac{2}{x-y} - \frac{3}{x}$

(o) $\frac{5}{x-2y} - \frac{3}{4x}$

2. Solve the following equations:

(a) $\frac{7}{2x} = \frac{5}{4}$

(b) $\frac{5}{x-3} = \frac{3}{7}$

(c) $\frac{x+5}{6} = \frac{3x-5}{8}$

(d) $\frac{x}{3} + 2x = 5$

(e) $\frac{12}{2x+1} = \frac{1}{x}$

(f) $\frac{x-1}{4} - \frac{2x-1}{6} = 3$

(g) $3 + \frac{2x+1}{3} = x$

(h) $\frac{4+x}{3} = \frac{2x-5}{5}$

(i) $\frac{1-2x}{4} + \frac{2-x}{2} = 4$

(j) $\frac{x+1}{4} - \frac{x}{5} = \frac{x}{6}$

(k) $\frac{x-1}{2} - \frac{x+1}{3} = x$

(l) $\frac{5}{4x} - \frac{3}{2x} + \frac{2}{x} = 5$

3. Make the letter in the brackets the subject of the formulae:

(a) $x = \frac{y}{y+1}$ (y)

(b) $a = \frac{1-t}{1+t}$ (t)

(c) $c = \frac{a-b}{a}$ (a)

(d) $x = \frac{3a+b}{a}$ (a)

(e) $a = p + \frac{prt}{100}$ (p)

(f) $k = h + \frac{2hk}{5}$ (h)

4. What is the number which when multiplied by 2 and added to 8 gives the same result as when it is divided by 2 and has 32 added to it?

5. When a number is divided by 4 and has 28 added to it, the result is equal to twice the number. Find the number.

6. The difference between the reciprocal of two consecutive positive even numbers is $\frac{1}{12}$. Find the two numbers.

7. One number is 3 times as large as the other and the difference between their reciprocals is $\frac{1}{6}$. Find the two numbers.

8. Given that $t = 2\pi r\sqrt{\frac{a^2+b^2}{10}}$, express a in terms of b, t and π.

9. If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{d}$, express b in terms of a, c and d.

10. The square of x is equal to the square root of y. Express y in terms of x.

11. A bus travels at 36 km/h and arrives at its destination half an hour late. If it travels at42 km/h, it arrives at the same destination half an hour earlier. Find the journey's distance.

12. A motorist does the first part of his journey atan average speed of 54 km/h. He then increases his speed to 60 km/h for the rest of the journey. If he travels 225 km in 4 hours, what is the distance travelled for the first part of his journey?

13. Mrs Lim buys 30 kg of cheese at a certain price. She finds that if she had bought some cheaper cheese costing 95 cents less per kg, she could have had $32\frac{1}{2}$kg for the same amount of money. What is the price per kg of the cheese which is more expensive?

14. \$$M$ is just sufficient to pay for the wages of one qualified teacher for x days or the wages of one relief teacher for y days. For how many days will \$$M$ be just sufficient to pay for the wages of one qualified teacher and one relief teacher? Give your answer in terms of x and y.

15. A man had a bag of sweets. He gave his son one sweet and $\frac{1}{7}$ of the remaining sweets. From what was left, he gave his daughter two sweets and $\frac{1}{7}$ of the remainder. The two children found that they had the same number of sweets. How many sweets were there initially in the bag?

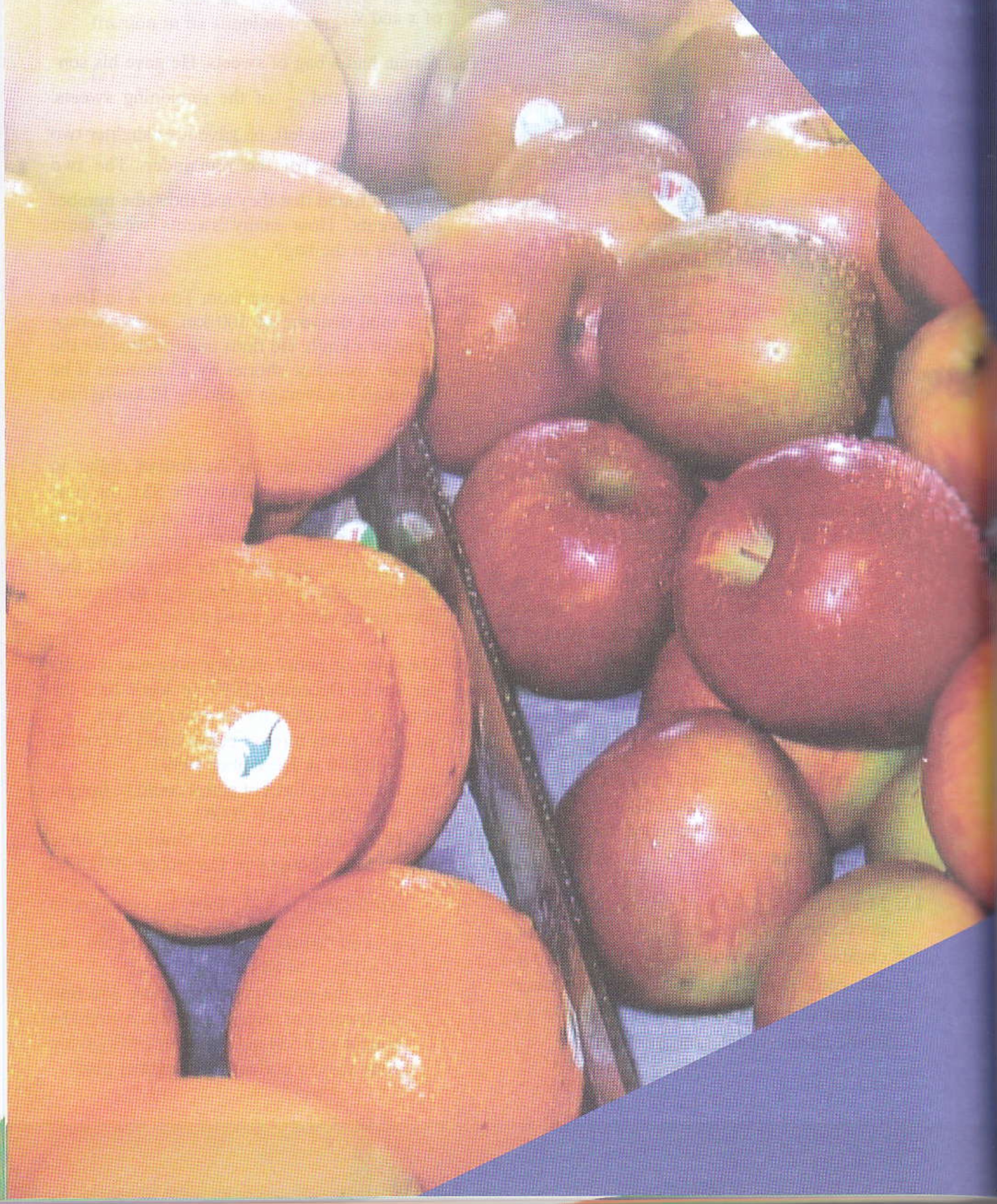
16. A man travels by a car from Town A to Town B, a total distance of 100 km, at an average speed of x km/h. He finds that the time for the journey would be shorter by 25 minutes if he increased his average speed by 12 km/h. Find x.

17. The exchange rate for Australian dollar in January 2000 was AU\$100 = S\$$x$. In June 2000, the exchange rate had become AU\$100 = S\$($x - 5$). Mr Chong found that he could get an extra AU\$32 for every S\$672 that he exchanged in June compared to January. Form an equation in x and solve it.

18. The price of petrol in Singapore was x cents per litre in December 1999. In April 2000 the price had increased by 12 cents per litre.
    (a) How many litres of petrol could be bought with \$58 in December 1999?
    (b) How many litres of petrol could be bought with \$58 in April 2000?
    (c) If the difference in the number of litres of petrol bought in December 1999 and April 2000 is $4\frac{11}{16}$, form an equation in x and show that it reduces to $x^2 + 12x - 14\,848 = 0$.
    (d) Solve the equation in (c) and use it to find the number of litres of petrol that could be bought with \$34 in December 1999, giving your answer correct to 1 decimal place.

In this chapter, you will learn to

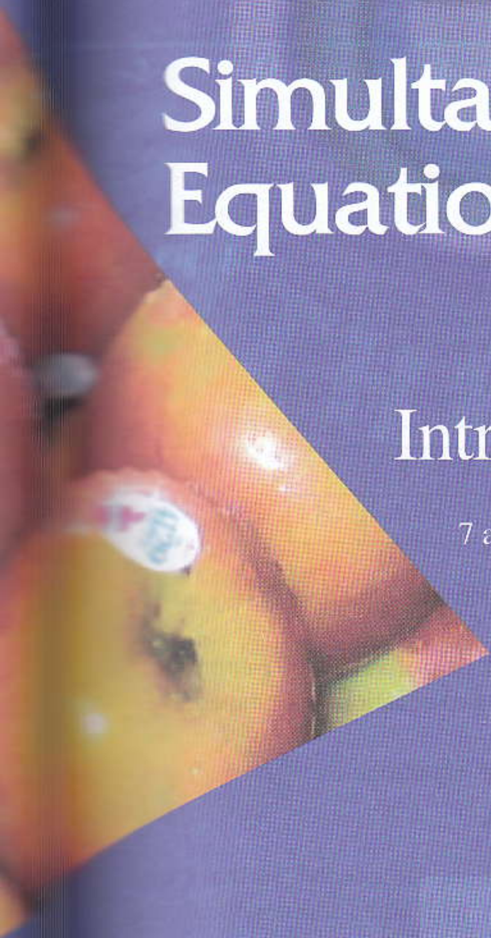
- *solve a pair of simultaneous linear equations by*
  (a) elimination,   (b) substitution;
- *apply the technique to solve some practical problems.*



CHAPTER

5

Simultaneous Linear Equations



Introduction

7 apples and 4 oranges cost \$4.10 while 5 apples and 7 oranges cost \$5.00. How much is each apple and each orange? We can solve this problem by solving simultaneous linear equations.

Simultaneous Linear Equations

In Book 1, we learnt how to solve linear equations with one variable such as $2x - 5 = 7$, $4y - 3 = 2y + 14$, $5t - 7 = 2t + 5$, etc. Each equation has one unique solution.

Let us now consider expressions with two unknowns such as $3x + 2y$.

If $x = 2$ and $y = 3$, then $3x + 2y = 3(2) + 2(3) = 12$.

If $x = 1$ and $y = -2$, then $3x + 2y = 3(1) + 2(-2) = -1$.

What is the value of $3x + 2y$ if the values of x and y are as follows?

(a) $x = 1, y = 2$
(b) $x = 3, y = -1$
(c) $x = 5, y = -4$
(d) $x = 7, y = -7$
(e) $x = -3, y = 8$

For the above pairs of values of x and y, the value of $3x + 2y$ is 7. In fact, there are infinitely many pairs of x and y that will satisfy $3x + 2y = 7$.

Let us consider another linear equation with two unknowns, $2x + y = 4$.

Examine the table of values below which uses different values of x and y for $2x + y$.

x	0	0	1	1	2	2	3	3	4	–1	–2	–3
y	3	4	1	2	0	–1	–1	–2	–4	–3	8	1
$2x + y$	3	4	3	4	4	3	5	4	4	–5	4	–5

Again, we see that there are several pairs of values of x and y which satisfy the equation $2x + y = 4$.

For the equations $3x + 2y = 7$ and $2x + y = 4$, check and see if there is any pair of values of x and y that satisfies both equations. Only one particular pair, $x = 1$ and $y = 2$ satisfies the two equations simultaneously.

We say that $x = 1$ and $y = 2$ is the solution to the simultaneous linear equations $3x + 2y = 7$ and $2x + y = 4$.

Thus, to solve a linear equation with two unknowns, two different equations are needed, so that each unknown has one unique solution. Similarly, to solve a linear equation with three unknowns, three different equations are needed, if each unknown is to have a unique solution.

There are three common methods of solving a pair of simultaneous equations. We shall introduce only two methods here: the **elimination method** and the **substitution method**. The third method, the **graphical solution method** will be dealt with in Chapter 8.

Solving Simultaneous Linear Equations Using Elimination Method

Let's solve the following pair of equations by the elimination method. We shall name the equations as equation (1) and equation (2).

$$3x - y = 12 \text{ ———— } (1)$$
$$2x + y = 13 \text{ ———— } (2)$$

The elimination method is usually used when the equations have the same variable term (disregarding the sign). Do equations (1) and (2) have the same variable term? What happens when we add up the equations?

$$(3x - y) + (2x + y) = 12 + 13$$

The terms in y cancel out. We are left with one unknown x in one linear equation.

$$5x = 25$$
$$x = 5$$

Substitute $x = 5$ into (1): $\quad 3(5) - y = 12$

$$y = 3$$

$\therefore \quad x = 5$ and $y = 3$ is the solution of the simultaneous equations.

Check by substituting $x = 5$, $y = 3$ into (1) and (2):
In (1), LHS $= 3(5) - 3 = 12 =$ RHS
In (2), LHS $= 2(5) + 3 = 10 + 3 = 13 =$ RHS.

> Your Attention
> *The checking step is important to make sure you get the correct answer.*

Example 1

Solve the simultaneous equations

$$3x + 7y = 17,\ 3x - 6y = 4.$$

Solution

$3x + 7y = 17$ ——— (1)
$3x - 6y = 4$ ——— (2)

If we subtract equation (2) from equation (1), the terms in x cancel out.

i.e., $(3x + 7y) - (3x - 6y) = 17 - 4$

$$13y = 13$$
$$y = 1$$

Substitute $y = 1$ into (1): $3x + 7(1) = 17$

$$3x = 10$$

$$x = \frac{10}{3}$$

$$= 3\frac{1}{3}$$

$\therefore$ the solution is $x = 3\frac{1}{3}$, $y = 1$.

Check:

Substitute $x = 3\frac{1}{3}$, $y = 1$ into (2),

$$3x - 6y = 3\left(3\frac{1}{3}\right) - 6(1) = 10 - 6 = 4$$

$\therefore$ LHS = RHS.

Problem Solving Tips

*To eliminate an unknown, the numerical value of the coefficient of the unknown in both the equations must be the same. If the signs of the coefficients are the **same**, we should **subtract**; if the signs are **different**, we should **add**.*

Example 2

Solve the simultaneous equations

$$4x - 5y = 17, x - 5y = 8$$

Solution

$4x - 5y = 17$ ______ (1)

$x - 5y = 8$ ______ (2)

We eliminate y as the numerical value of the coefficient of y is the same. We subtract equation (2) from (1) as the signs are the same.

(1) – (2) gives $(4x - 5y) - (x - 5y) = 17 - 8$

$$3x = 9$$

$$\therefore \quad x = 3$$

Substitute $x = 3$ into (1): $4(3) - 5y = 17$

$$-5y = 5$$

$$\therefore \quad y = -1$$

$\therefore$ the solution is $x = 3$, $y = -1$.

Check:

Substitute $x = 3$, $y = -1$ into (2)

$$4x - 5y = 4(3) - 5(-1)$$

$$= 12 + 5 = 17$$

$\therefore$ LHS = RHS.

Exercise 5a

1. Solve the following simultaneous equations:
   (a) $x + y = 3$
   $3x + y = 5$
   (b) $2x + 3y = 7$
   $5x + 3y = 13$
   (c) $3x + 2y = 5$
   $7x + 2y = 9$
   (d) $x + 5y = 8$
   $x + 3y = 6$
   (e) $4x + 3y = 13$
   $7x + 3y = 16$
   (f) $3x + 8y = 13$
   $3x + 2y = 1$
   (g) $9x + 5y = 19$
   $2x + 5y = 12$
   (h) $7x + 2y = 19$
   $7x + 8y = 13$
   (i) $11x + 3y = -10$
   $11x + 7y = 6$
   (j) $8x + 13y = 2$
   $5x + 13y = 11$

2. Solve the following simultaneous equations:
   (a) $x - y = 3$
   $4x + y = 17$
   (b) $4x + y = 11$
   $3x - y = 3$
   (c) $3p + q = 14$
   $5p - q = 26$
   (d) $5p - 2q = 9$
   $3p + 2q = 7$
   (e) $3h + 4k = 1$
   $5h - 4k = 7$
   (f) $3p + 2q = 9$
   $3p - 2q = 3$
   (g) $5x - 3y = 6$
   $7x + 3y = 12$
   (h) $6x + 7y = 2$
   $2x - 7y = 10$
   (i) $13x + 9y = 4$
   $17x - 9y = 26$
   (j) $11x + 4y = 12$
   $9x - 4y = 8$

3. Solve the following simultaneous equations:
   (a) $x + y = 16$
   $x - y = 0$
   (b) $x - y = 5$
   $x + y = 19$
   (c) $x - y = 6$
   $x + y = 12$
   (d) $2x + y = 23$
   $4x - y = 19$
   (e) $3x + 2y = 13$
   $3x - 2y = 5$
   (f) $5x - 2y = 12$
   $3x + 2y = 12$
   (g) $4x - y = 7$
   $4x + 3y = 11$
   (h) $3x - 2y = 9$
   $2x - 2y = 7$
   (i) $5x - 6y = 14$
   $5x - 5y = 15$
   (j) $7x - 3y = 15$
   $11x - 3y = 21$

4. Solve the following simultaneous equations:
   (a) $3x - 2y = 5$
   $2y - 5x = 9$
   (b) $2x + 3y = 10$
   $3y - x = 7$
   (c) $3x - 5y = 13$
   $5y + x = 7$
   (d) $6x - y = 23$
   $3y + 6x = 11$
   (e) $7x + 2y = 33$
   $3y - 7x = 17$
   (f) $3x - y + 14 = 0$
   $2x + y + 1 = 0$
   (g) $3y - 2x + 15 = 0$
   $2x - 2y + 19 = 0$
   (h) $5x + 7y - 17 = 0$
   $7y + 3x - 27 = 0$

More Examples on Elimination

Sometimes it is necessary to change the coefficients of the unknowns in one of the equations before we can eliminate an unknown by addition or subtraction.

Example 3

Solve the simultaneous equations

$$3x + 2y = 8, \; 4x - y = 7$$

Solution

$3x + 2y = 8$ ——— (1)
$4x - y = 7$ ——— (2)

Multiply (2) by 2 to make the coefficients of y in both equations numerically equal.

$$8x - 2y = 14 \text{ ——— (3)}$$

(1) + (3): $(3x + 2y) + (8x - 2y) = 8 + 14$

$$11x = 22$$

$$x = 2$$

Substitute $x = 2$ into (2): $4(2) - y = 7$

$$\therefore \; y = 1$$

$\therefore$ the solution is $x = 2$, $y = 1$.

Just For Fun

Under the law of a certain country, anyone caught poaching in the King's private forest is punished by being hanged or beheaded. However, before the culprits are hanged or beheaded, they have to make a statement. If the statement is true, they will be beheaded; if it is false, they will be hanged.

One day, a man was caught poaching in the King's private forest and the statement he made was:

"I shall be hanged."

Now, this is where the problem lies. If he was hanged, then his statement would be true and therefore he should be beheaded.

But if he was beheaded, then what he had said was true. But he did not say that he would be beheaded. Should he be hanged or be beheaded? This type of problem is called a logical paradox.

A paradox is a statement which contradicts itself.

You may wish to read more about logical paradoxes in your library or from the internet.

Sometimes it is necessary to change the coefficients of the unknowns in both of the equations before we can eliminate an unknown.

Example 4

Solve the simultaneous equations

$$13x - 6y = 20, \ 7x + 4y = 18.$$

Solution

$13x - 6y = 20$ ——— (1)

$7x + 4y = 18$ ——— (2)

The coefficients of y in both equations will be numerically equal if we multiply (1) by 2 and (2) by 3.

Notes

The LCM of 6 and 4 is 12.

(1) × 2: $26x - 12y = 40$ ——— (3)

(2) × 3: $21x + 12y = 54$ ——— (4)

(3) + (4): $47x = 94$

$x = 2$

Substitute $x = 2$ into (1): $13(2) - 6y = 20$

$6 = 6y$

$y = 1$

$\therefore$ the solution is $x = 2$, $y = 1$.

Do you think it will be easier if we eliminate x first? Why?

In the process of solving simultaneous linear equations, either one of the unknowns may be eliminated first.

Example 5

Solve the simultaneous equations

$$\frac{2x}{3} - \frac{y}{9} = 6, x - \frac{y}{3} = 6.$$

Solution

$$\frac{2x}{3} - \frac{y}{9} = 6 \quad \text{——— (1)}$$

$$x - \frac{y}{3} = 6 \quad \text{——— (2)}$$

$(1) \times \frac{3}{2}$: $$x - \frac{y}{6} = 9 \quad \text{——— (3)}$$

(3) – (2): $$\left(x - \frac{y}{6}\right) - \left(x - \frac{y}{3}\right) = 9 - 6$$

$$-\frac{y}{6} + \frac{y}{3} = 3$$

$$\frac{y}{6} = 3$$

$$y = 18$$

Substitute $y = 18$ into (2): $$x - \frac{18}{3} = 6$$

$$x = 12$$

$\therefore$ the solution is $x = 12$, $y = 18$.

Notes

Alternatively, we can eliminate the fractional part by multiplying (1) by 9 and (2) by 3.

$(1) \times 9: 6x - y = 54$ —(3)
$(2) \times 3: 3x - y = 18$ —(4)
$(3) - (4): \quad 3x = 36$
$x = 12$
Substitute $x = 12$ *into* (4):
$3(12) - y = 18$
$y = 18$
$\therefore$ *the solution is*
$x = 12, y = 18.$

Exercise 5b

1. Solve the following simultaneous equations.

(a) $2x + 3y = 7$
$3x + y = 7$

(b) $4x + y = 11$
$3x + 2y = 7$

(c) $x + 2y = 3$
$3x + 5y = 7$

(d) $3x - y = 17$
$7x - 3y = 41$

(e) $7x - 3y = 13$
$2x - y = 3$

(f) $4x - 3y = 2$
$x - 5y = 9$

(g) $9x - 5y = 2$
$3x - 4y = 10$

(h) $5x - 3y = 13$
$7x - 6y = 20$

(i) $7h - 2k = 17$
$3h + 4k = 17$

(j) $4x - 3y = 31$
$16x + 5y = 39$

2. Solve the following simultaneous equations.

(a) $2x + 3y = 8$
$5x + 2y = 9$

(b) $4x - 3y = -1$
$5x - 2y = 4$

(c) $6x + 7y = 25$
$7x - 3y = 18$

(d) $5x + 4y = 11$
$3x + 5y = 4$

(e) $9x + 8y = -3$
$5x - 3y = 43$

(f) $9x + 2y = 5$
$7x - 3y = 13$

(g) $5x - 4y = 17$
$2x - 3y = 11$

(h) $7x + 3y = 8$
$3x - 4y = 14$

(i) $10x - 3y = 24.5$
$3x - 5y = 13.5$

(j) $6x + 5y = 10.5$
$5x - 3y = -2$

3. Solve the following simultaneous equations:

(a) $13 + 2y = 9x$
$3y = 7x$

(b) $2x - 3y = 5$
$3x - \frac{2y - 3}{5} = 4$

(c) $3x - y = 23$
$\frac{x}{3} + \frac{y}{4} = 4$

(d) $2x - 3y = 24$
$\frac{5x}{3} - \frac{y}{2} = 12$

(e) $\frac{5x}{8} + \frac{7y}{18} = 6$
$3x - y = 3$

(f) $\frac{x - 3}{5} = \frac{y - 7}{2}$
$11x = 13y$

(g) $\frac{1}{5}(x - 2) = \frac{1}{4}(1 - y)$
$26x + 3y + 4 = 0$

(h) $\frac{1}{3}(x + y) = \frac{1}{5}(x - y)$
$3x + 11y = 4$

(i) $\frac{1}{5}(x + y) = \frac{1}{7}(x - y)$
$3x + 17y = 2$

(j) $4(2x - y + 3) = 0$
$2(x + y) - 3(x - y) = 6$

Solving Simultaneous Linear Equations Using Substitution Method

Let's solve the following pair of simultaneous equations

$$7x - 2y = 21 \quad \text{——— (1)}$$

$$4x + y = 57 \quad \text{——— (2)}$$

Make y the subject of equation (2): $y = 57 - 4x$ ——— (3)

Substitute (3) into (1):

$$\begin{aligned} 7x - 2(57 - 4x) &= 21 \\ 7x - 114 + 8x &= 21 \\ 15x &= 114 + 21 \\ 15x &= 135 \\ x &= 9 \end{aligned}$$

Substitute $x = 9$ into (3):

$$\begin{aligned} y &= 57 - 4(9) \\ &= 57 - 36 \\ &= 21 \end{aligned}$$

$\therefore$ the solution is $x = 9$, $y = 21$.

The method we have used here to solve simultaneous equations is called the **substitution method**.

If we make x the subject of the equation (1) or (2) in the above example, will we get the same solution? Which way is easier?

Example 6

Solve the simultaneous equations

$$3x + 2y = 7,\ 9x + 8y = 22.$$

Solution

$$3x + 2y = 7 \quad \text{——— (1)}$$

$$9x + 8y = 22 \quad \text{——— (2)}$$

From (1), $2y = 7 - 3x$

$$y = \frac{7 - 3x}{2} \quad \text{——— (3)}$$

Substitute (3) into (2): $9x + 8\left(\frac{7 - 3x}{2}\right) = 22$

$$9x + 28 - 12x = 22$$

$$-3x = -6$$

$$x = 2$$

Substitute $x = 2$ into (3): $y = \frac{7 - 3(2)}{2} = \frac{1}{2}$

$\therefore$ the solution is $x = 2$, $y = \frac{1}{2}$.

Do you think it will be easier to solve this question by the elimination method?

Thinking Time

Consider the following simultaneous equations:

$$2x + y = 6 \quad \text{——— (1)}$$

$$x = 1 - \frac{1}{2}y \quad \text{——— (2)}$$

Substitute (2) into (1);

$$2(1 - \frac{1}{2}y) + y = 6$$

$$2 - y + y = 6$$

$$\therefore \quad 2 = 6$$

Do you know where the problem lies?

Exercise 5c

1. Use the substitution method to solve the following simultaneous equations:

   (a) $x + y = 7$
   $x - y = 5$

   (b) $3x - y = 0$
   $2x + y = 5$

   (c) $2x - 7y = 5$
   $3x + y = -4$

   (d) $5x - y = 5$
   $3x + 2y = 29$

   (e) $5x + 3y = 11$
   $4x - y = 2$

   (f) $5x - 2y = 4$
   $x + y = 9$

   (g) $x + y = 13$
   $3x - 5y = 7$

   (h) $5x + 2y = 3$
   $x - 4y = -6$

   (i) $3x - 2y = 8$
   $4x + 3y = 5$

   (j) $2x + 5y = 12$
   $4x + 3y = -4$

2. Solve the following simultaneous equations by using either the elimination or the substitution method.

   (a) $2x - y = 1$
   $5x + 3y = 30$

   (b) $4x - y = 6$
   $8x + y = 24$

   (c) $6x + 5y = 9$
   $4x - 3y = 25$

   (d) $2y - 5x = 25$
   $4x + 3y = 3$

   (e) $3x + 7y = 2$
   $6x - 5y = 4$

   (f) $\frac{1}{5}x + y + 2 = 0$
   $\frac{1}{3}x - y - 10 = 0$

   (g) $\frac{x + y}{3} = 3$
   $\frac{3x + y}{5} = 1$

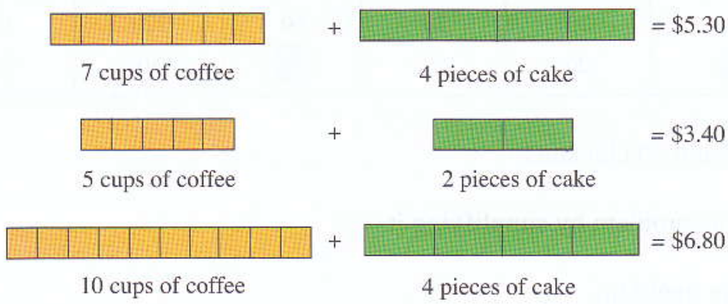
   (h) $\frac{x}{3} + \frac{y}{2} = 4$
   $\frac{2x}{3} - \frac{y}{6} = 1$

Problem Solving Involving Simultaneous Equations

Many mathematical and real-life problems can be solved by using the technique of solving simultaneous equations.

On two occasions, Mr Muthu went to a coffee shop with some friends. He was very generous and gave his friends treats. On the first occasion, 7 cups of coffee and 4 pieces of cake cost him \$5.30. On the second occasion, 5 cups of coffee and 2 pieces of cake cost him \$3.40.
A friend worked out the cost of one cup of coffee and one piece of cake by drawing a diagram.

Strategy 1: Draw a diagram



$\therefore$ the extra 3 cups of coffee cost \$6.80 – \$5.30 = \$1.50

$\therefore$ 1 cup of coffee costs \$0.50.

and 5(\$0.50) + cost of 2 pieces of cake = \$3.40

cost of 2 pieces of cake = \$3.40 – \$2.50 = \$0.90

$\therefore$ 1 piece of cake costs \$0.45.

Another friend worked out the cost of one cup of coffee and one piece of cake by using the algebraic method.

Strategy 2: Use algebraic method

Let \$$x$ be the cost of one cup of coffee and \$$y$ be the cost of one piece of cake.

$$7x + 4y = 5.3 \quad \text{——— (1)}$$

$$5x + 2y = 3.4 \quad \text{——— (2)}$$

(2) × 2: $$10x + 4y = 6.8 \quad \text{——— (3)}$$

(3) – (1): $$3x = 1.5$$

$$\therefore \quad x = 0.5$$

Substitute $x = 0.5$ into (2):

$$5(0.5) + 2y = 3.4$$
$$2y = 3.4 - 2.5$$
$$= 0.9$$
$$y = 0.45$$

Hence one cup of coffee costs \$0.50 and one piece of cake costs \$0.45.

A farmer keeps some chickens and goats on his farm. One day, his three sons want to know how many animals there are on the farm. The farmer tells them that there are altogether 50 heads and 140 legs of animals. How can his sons work out the number of goats and chickens?

The first son finds out the answer by making a systematic list.

Strategy 1: Make a systematic list

Number of goats	*Number of chickens*	*Number of heads*			*Number of legs*		
		Goats	*Chickens*	*Total*	*Goats*	*Chicken*	*Total*
50	0	50	0	50	200	0	200
40	10	40	10	50	160	20	180
30	20	30	20	50	120	40	160
20	30	20	30	50	80	60	140

Thus, there are 20 goats and 30 chickens.

The second son solves the problem by simplifying it.

Strategy 2: Simplify the problem

Imagine that all the goats stand on their hind legs. There are 50 heads counted. When the goats stand on their hind legs, there would also be 50 pairs of legs on the ground. But 140 legs or 70 pairs of legs are counted. Therefore, the number of goat legs in the air must be $70 - 50 = 20$ pairs.

Hence, there must be 20 goats.

$\therefore$ there are $50 - 20 = 30$ chickens.

The third son solves the problem by using the algebraic method.

Strategy 3: Use the algebraic method

Let x be the number of goats and y be the number of chickens.

$$x + y = 50 \quad \text{———} \quad (1)$$
$$4x + 2y = 140 \quad \text{———} \quad (2)$$

(1) × 2: $2x + 2y = 100$ ——— (3)

(2) – (3): $2x = 40$

$x = 20$

and $20 + y = 50$

$\therefore \quad y = 30$

$\therefore$ there are 20 goats and 30 chickens.

Which method do you prefer?

Example 7

Find two numbers whose sum is 67 and whose difference is 3.

Solution

Let x be the greater number and y be the smaller number.

$$x + y = 67 \quad \text{\_\_\_\_\_\_ (1)}$$
$$x - y = 3 \quad \text{\_\_\_\_\_\_ (2)}$$

(1) + (2):
$$2x = 70$$
$$x = 35$$

Substitute $x = 35$ into (1):
$$35 + y = 67$$
$$y = 67 - 35$$
$$= 32$$

$\therefore$ the two numbers are 32 and 35.

Thinking Time

Can you solve the above problem using only one unknown x?

Example 8

If 1 is added to the numerator and 2 to the denominator of a fraction, its value will be $\frac{2}{3}$. If 2 is subtracted from the numerator and 1 from the denominator, its value will be $\frac{1}{3}$. What is the fraction?

Solution

Let x be the numerator and y be the denominator of the fraction, i.e., let the fraction be $\frac{x}{y}$.

$$\frac{x+1}{y+2} = \frac{2}{3} \quad \text{\_\_\_\_\_\_ (1)}$$

$$\frac{x-2}{y-1} = \frac{1}{3} \quad \text{\_\_\_\_\_\_ (2)}$$

By clearing the fractions and simplifying, we get

$$3x - 2y = 1 \quad \text{——— (3)}$$
$$3x - y = 5 \quad \text{——— (4)}$$

(3) – (4): $-y = -4$

$y = 4$

Substitute $y = 4$ into (3): $3x - 2(4) = 1$

$3x = 9$

$x = 3$

$\therefore$ the fraction is $\frac{3}{4}$.

Example 9

The sum of the ages of Mr. Singh and his son is 60 years. Two years ago, Mr. Singh was three times as old as his son. How old is Mr. Singh's son? How old was Mr. Singh when his son was born?

Solution

Let Mr. Singh's age be x years and his son's age be y years.

We have $x + y = 60$ ——— (1)

Two years ago, Mr. Singh's age was $(x - 2)$ years old and his son's age was $(y - 2)$ years.

$\therefore \quad x - 2 = 3(y - 2)$

$x - 2 = 3y - 6$

$\therefore \quad x = 3y - 4$ ——— (2)

Substituting (2) into (1)

$(3y - 4) + y = 60$

$\therefore \quad 4y = 64$

$y = 16$

$\therefore$ Mr. Singh's son is 16 years old.

Substituting $y = 16$ into (2)

$$x = 3(16) - 4 = 48 - 4 = 44$$

$\therefore$ Mr. Singh was $(44 - 16)$ years old, i.e., 28 years old.

Example 10

The sum of the digits of a two-digit number is 8. When the number with the same digits reversed is subtracted from the number, the difference is 18. What is the number?

Solution

Let x be the *tens* digit and y be the *units* digit.

The number is $10x + y$.

The reversed number is $10y + x$.

$$x + y = 8 \quad \text{——— (1)}$$
$$10x + y - (10y + x) = 18$$
$$10x + y - 10y - x = 18$$
$$9x - 9y = 18 \quad \text{——— (2)}$$

(2) ÷ 9: $\quad x - y = 2 \quad$ ——— (3)

(1) + (3): $\quad x + y + x - y = 8 + 2$

$$2x = 10$$
$$x = 5$$

Substitute $x = 5$ into (1): $\quad 5 + y = 8$

$$y = 8 - 5$$
$$= 3$$

∴ the required number is 53.

Exercise 5d

1. Find two numbers whose sum is 138 and whose difference is 88.

2. One third of the sum of two angles is 60° and one quarter of their difference is 28°. Find the two angles.

3. The sum of two numbers is 36 and their difference is 9. Find the two numbers.

4. One fifth of the sum of two angles is 24° and half their difference is 14°. Calculate the two angles.

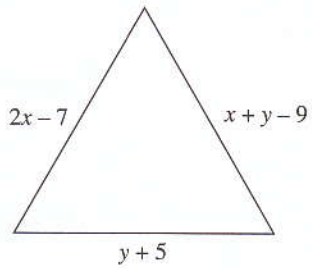
5. 8 kg of potatoes and 5 kg of carrots cost \$28 whereas 2 kg of potatoes and 3 kg of carrots cost \$11.20. What is the cost of 1 kg of each item?

6. 6 stools and 4 chairs cost \$58 but 5 stools and 2 chairs cost \$35. Find the cost of each stool and each chair.

7. Adding unity to the numerator as well as the denominator of a fraction makes it equal to $\frac{4}{5}$. Subtracting 5 from each makes it equal to $\frac{1}{2}$. What is the fraction?

8. A belt and a wallet cost \$42 while 7 belts and 4 wallets cost \$213. Calculate the cost of each item.

9. The figure shown is an equilateral triangle. Calculate the length of each side and give your answer in cm.

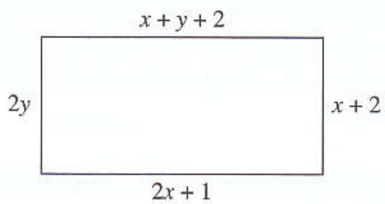


10. A fraction equals $\frac{1}{2}$ if 1 is subtracted from both the numerator and denominator. It is equal to $\frac{2}{3}$ if 1 is added to both the numerator and denominator. Find the fraction.

11. Find two numbers given that their sum is 48 and the smaller number is equal to one-fifth of the larger number.

12. \$80 is divided between two men such that one-quarter of one person's share is equal to $\frac{1}{6}$ of the other. How much will each man receive?

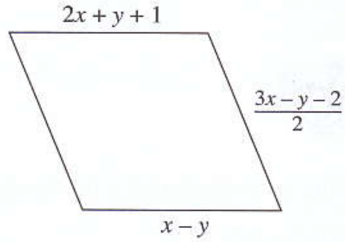
13. Find the perimeter, in cm, of the rectangle shown below.



14. I think of a pair of numbers. If I add 7 to the first, I obtain a number which is twice the second. If I add 20 to the second, I obtain a number which is four times the first. What are the numbers?

15. The figure shown is a rhombus. Its measurements are in cm. Calculate

   (a) the length of a side,
   (b) the perimeter of the figure.



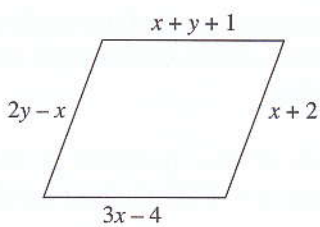
16. I have fifteen coins. Some are valued at 5 cents and others at 20 cents. Their total value is \$1.65. How many 5-cent coins and 20-cent coins do I have?

17. In five years' time, a father will be three times as old as his son. Four years ago, the father was six times as old as his son. Find their present ages.

18. A two-digit number is such that the sum of the digits is 11. When the number with the same digits reversed is subtracted from this number, the difference is 9. What is the number?

19. The difference between two numbers is 10 and their sum is equal to four times the smaller number. What are the numbers?

20. The sides of the parallelogram shown in the figure are given in centimetres. Find x and y. Hence, find the perimeter of the parallelogram.



21. The sum of two numbers is 40. If 2 is added to the larger number, the result is equal to twice the smaller number. What are the two numbers?

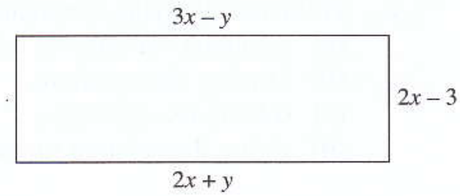
22. A man buys 36 stamps, all of which are either 50¢ or 10¢ pieces and the total value is \$11.60. How many stamps of each kind does he buy?

23. The length of a rectangle is greater than its breadth by 2 cm. If the length is increased by 4 cm and the breadth decreased by 3 cm, the area remains the same. Find the length and breadth of the rectangle.

24. \$2500 is to be deposited in Banks A and B. The interest rate per annum for Bank A is 6% while that for Bank B is $6\frac{1}{2}\%$. After one year, their interests are equal. How much money is to be deposited in each bank?

25. A train takes 7 hours to travel from Station A to Station B. If its speed is increased by 10 km/h, it will take 1 hour less to complete the journey. Find the distance between the two stations.

26. The figure below shows the lengths of the sides of a rectangle in cm. Given the perimeter of the rectangle is 120 cm, find the values of x and y and then the area of the rectangle.



Explorations

We learnt that to solve a pair of simultaneous equations involving two unknowns, we need to have two equations. Similarly, to solve three simultaneous equations involving three unknowns, we will need to have three equations. Can you solve a pair of simultaneous equations with three unknowns?

Try to solve the following equations:

1. A rooster costs \$5 and a hen costs \$3. Chicks are sold at 3 for a dollar. A farmer bought 100 birds of these three types for \$100. How many of each type of bird did he buy?
(There are three possible sets of answers.)

2. A rabbit costs \$3.50, ducklings are sold at 3 for a dollar and chicks cost 50 cents each. A farmer paid \$100 for 100 of these animals. How many of each did he buy?
(There are five possible sets of answers.)

SUMMARY

1. A pair of simultaneous linear equations in two variables can be solved by
   - **(a)** elimination method,
   - **(b)** substitution method,
   - **(c)** graphical method.

2. Problems involving simultaneous equations may be solved by
   - **(a)** assigning variables to unknowns,
   - **(b)** forming the equations,
   - **(c)** solving the equations,
   - **(d)** giving the solution to the problem.

Review Examples 5

Example 1

The ratio of Jane's age to Peter's age is 5:7. In six years' time, the ratio of their ages will become 4:5. What will the ratio of their ages be in 30 years' time?

Solution

Let the ages of Jane and Peter be x and y years old.

$\therefore$ we have $x : y = 5 : 7$

i.e. $\frac{x}{y} = \frac{5}{7}$ ——— (1)

In 6 years' time, Jane will be $(x + 6)$ years old and Peter will be $(y + 6)$ years old.

$\therefore \quad \frac{x+6}{y+6} = \frac{4}{5}$ ——— (2)

From (1), $x = \frac{5}{7}y$ ——— (3)

Just for fun

Spiders, dragonflies and houseflies are kept in three separate containers. It is found that there are 20 heads, 136 legs and 19 pairs of wings altogether in the three containers. Find the number of spiders, dragonflies and houseflies given that a spider has 8 legs, a dragonfly has 6 legs and 2 pairs of wings and a housefly has 6 legs and 1 pair of wings.

Substitute (3) into (2):

$$\frac{\frac{5}{7}y+6}{y+6}=\frac{4}{5}$$

Multiplying both sides by $5(y + 6)$, we have

$$5\left(\frac{5}{7}y+6\right) = 4(y + 6)$$

$$\therefore \quad \frac{25}{7}y + 30 = 4y + 24$$

$$\frac{3}{7}y = 6$$

$$y = 14$$

Substitute $y = 14$ into (3): $x = \frac{5}{7}(14) = 10$.

$\therefore$ in 30 years' time, Jane will be $(10 + 30)$ years old and Peter will be $(14 + 30)$ years old.

$\therefore$ the ratio of their ages $= 40 : 44 = 10 : 11$.

Example 2

Solve the simultaneous equations

$$\frac{4}{x}+\frac{15}{y}=15,\ \frac{7}{5x}-\frac{6}{y}=3$$

Solution

Let $a = \frac{1}{x}$ and $b = \frac{1}{y}$. Then

$$\frac{4}{x}+\frac{15}{y} = 15 \longrightarrow 4a + 15b = 15 \text{ —— (1)}$$

and $$\frac{7}{5x}-\frac{6}{y} = 3 \longrightarrow \frac{7}{5}a - 6b = 3 \text{ —— (2)}$$

(2) × $\frac{5}{2}$: $$\frac{7}{2}a - 15b = \frac{15}{2} \text{ —— (3)}$$

(1) + (3): $$\frac{15}{2}a = 22\frac{1}{2}$$

$$a = 3$$

Problem Solving Tips

The idea of converting a non-linear equation to a linear one by substitution is very important in problem solving.

That is $\frac{1}{x} = 3$

$\therefore x = \frac{1}{3}$

Substitute $a = 3$ into (2):

$$\frac{7}{5}(3) - 6b = 3$$

$$6b = 1\frac{1}{5}$$

$$b = \frac{1}{5}$$

That is $\frac{1}{y} = \frac{1}{5}$

$\therefore y = 5.$

$\therefore$ the solution is $x = \frac{1}{3}, y = 5.$

Review Questions 5

1. Solve the following simultaneous equations:
   - **(a)** $7x + 2y = 10$
     $5x + 2y = 6$
   - **(b)** $9x + 4y = 28$
     $4y - 11x = -12$
   - **(c)** $2x - 5y = 22$
     $2x - 3y = 14$
   - **(d)** $6x - y = 16$
     $3x + 2y = -12$
   - **(e)** $x + y = 0.5$
     $x - y = 1$
   - **(f)** $4x + 3y = 0$
     $5y + 53 = 11x$
   - **(g)** $8x + 3y = 14$
     $2x + y = 4$
   - **(h)** $2x + 0.4y = 8$
     $5x - 1.2y = 9$
   - **(i)** $5x - 4y = 4$
     $2x - y = 2.5$

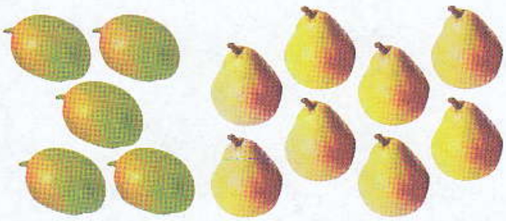
2. 5 apples and 4 oranges cost \$3.40 while 7 apples and 6 oranges cost \$4.90. Find the cost of an apple and an orange.

3. I think of a pair of numbers. If I add 11 to the first, I obtain a number which is twice the second. If I add 20 to the second, I obtain a number which is twice the first. What are the numbers?

4. If *A* gives *B* \$3, *B* will have twice as much as *A*. If *B* gives *A* \$5, *A* will have twice as much as *B*. How much does each have?

5. If the larger of two numbers is divided by the smaller, the quotient and the remainder are 2 each. If 5 times the smaller number is divided by the larger, the quotient and the remainder are still 2 each. Find the two numbers.

6. If the selling price of 5 pears and 4 mangoes is \$1.75 while that of 8 pears and 5 mangoes is \$2.45, what is the price of each pear and each mango?



7. At a basketball game, adult tickets were sold at \$1.00 each and student tickets at 75¢ each. If 150 tickets were sold and \$140 was collected, how many tickets of each kind were sold?

8. The sum of the numerator and denominator of a fraction is 17. If 3 is added to the numerator, the value of the fraction will be 1. What is the fraction?

9. The denominator of a certain fraction exceeds its numerator by 4. If $\frac{2}{3}$ is added to the reciprocal of the fraction, the sum becomes 3. Find the fraction.

10. In four years' time, a father will be three times as old as his son. Six years ago, he was seven times as old as his son. How old are they now?

11. A shopkeeper mixed coffee powder worth \$2.50 per kg with coffee powder worth \$3.50 per kg, and sold 20 kg of the mixture at \$2.80 per kg. Find the weights of the 2 grades of coffee powder that he mixed together.

12. Two cars leave a town at the same time and travel in opposite directions. The speed of one car is 12 km/h more than the other. They are 444 km apart after 3 hours. Find the speed of each car.

13. A motorist drove for 2 hours at one speed and then for 3 hours at another speed. He covered a distance of 252 km. If he had travelled 4 hours at the first speed and one hour at the second speed, he would have covered 244 km. Find the two speeds.

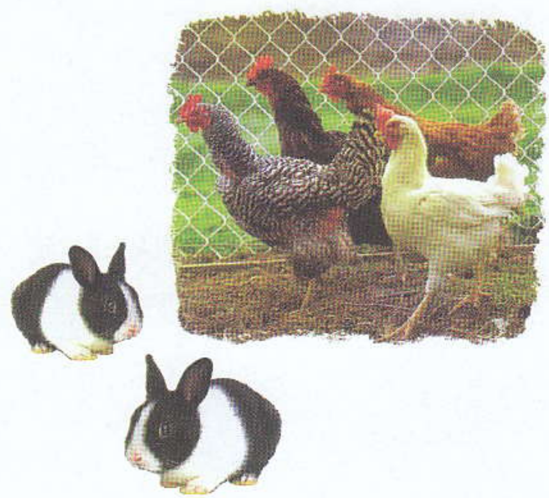
14. The sum of the digits of a two-digit number is 12, and the *units* digit is twice the *tens* digit. Find the number.

15. Solve the following simultaneous equations:

    (a) $\frac{5}{x} - \frac{6}{y} = 1, \ \frac{17}{x} + \frac{30}{y} = 16$

    (b) $\frac{2}{x+y} = \frac{1}{2x+y}, \ 3x + 4y = 9$

16. A farmer keeps hens and rabbits on his farm. One day, he counted a total of 70 heads and 196 legs. How many more hens than rabbits does he have?



In this chapter, you will learn

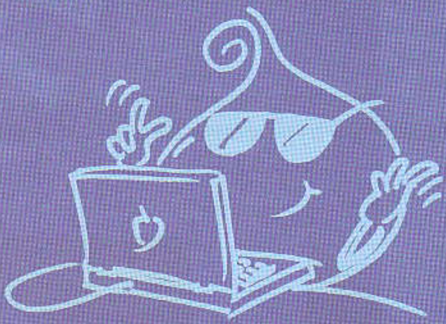
- ***Pythagoras' theorem;***
- ***how to find the length of a side of a right-angled triangle using Pythagoras' theorem;***
- ***how to solve problems involving Pythagoras' theorem.***

CHAPTER

6

Pythagoras' Theorem

Introduction



The picture shows an interior designer busily doing some calculations on the details of his drawing. He will need to make use of Pythagoras' theorem in the course of his work.

Do you know what's the Pythagoras' Theorem?

Pythagoras' Theorem

Fig. 6.1 shows a right-angled triangle with $B\hat{A}C = 90°$. The side opposite the right angle A is called the **hypotenuse**. It is the longest side of a right-angled triangle.

> **Notes**
> *We always denote the side opposite $\hat{A}$ by a, the side opposite $\hat{B}$ by b and the side opposite $\hat{C}$ by c.*

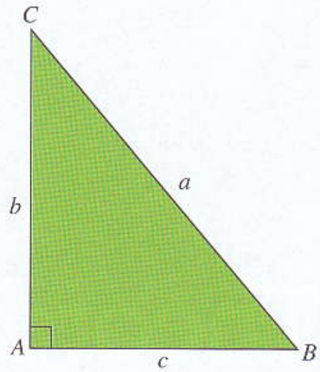
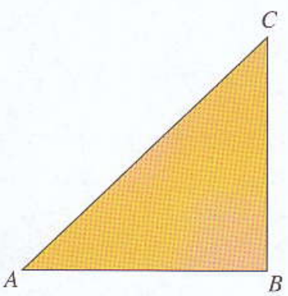
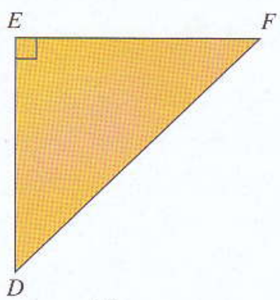


Fig. 6.1

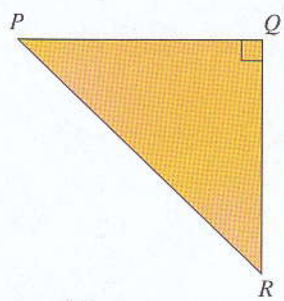
Can you identity the hypotenuse for each of the following triangles?



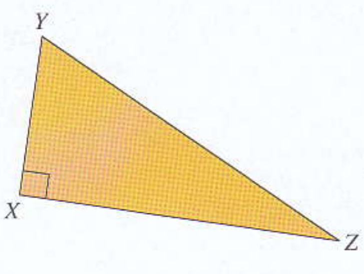
(a)



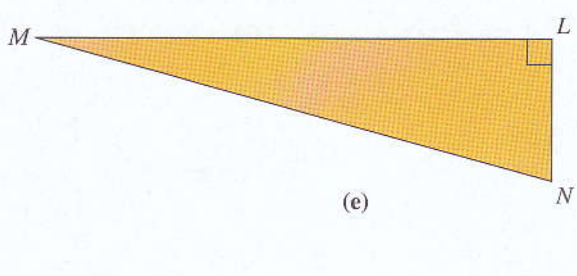
(b)



(c)



(d)



(e)

Explorations

Fig. 6.2 shows three right-angled triangles with the lengths of the two shorter sides given.

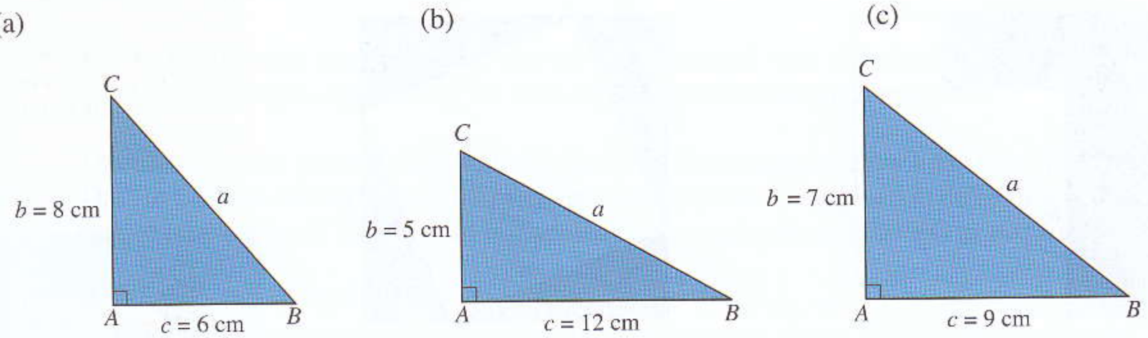


Fig. 6.2

Construct the three triangles in Fig. 6.2 accurately, measure the length of a for each triangle and fill in the following table.

	b	c	a	b^2	c^2	a^2	$b^2 + c^2$
(a)	8	6		64	36		
(b)	5	12		25	144		
(c)	7	9		49	81		

Do you notice the relation between a^2 and $b^2 + c^2$?
Draw more right-angled triangles and repeat the above. Do you get the same conclusion?

From the above table we find that $a^2 = b^2 + c^2$. This important relation between the three sides of a right-angled triangle was discovered by a famous Greek philosopher and mathematician, Pythagoras, in 6th century BC. In honour of his great contribution, this relation is named **Pythagoras' Theorem** which states that:

The square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other two sides.

i.e., $a^2 = b^2 + c^2$ or $BC^2 = AC^2 + AB^2$ as in Fig. 6.1.

There are more than 300 proofs of Pythagoras' Theorem. We shall examine one of them and learn how to apply it to solve problems.

Take eight right-angled cardboard triangles of the same size and arrange them in the two designs shown below:

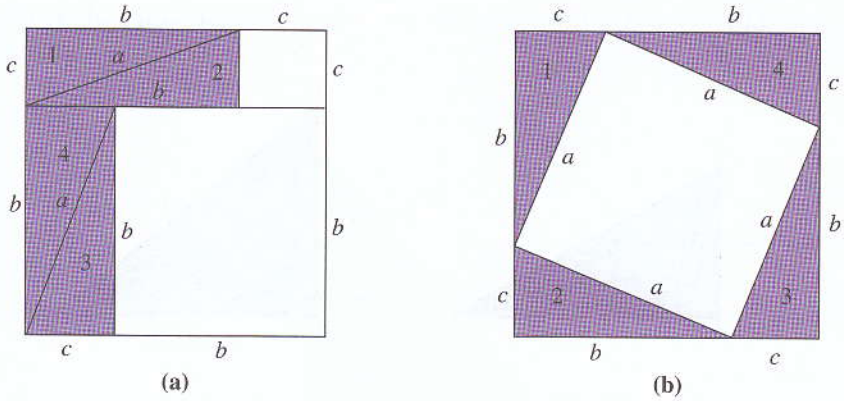


Fig. 6.3

Fig. 6.3(a) and Fig. 6.3(b) are squares of sides $(b + c)$ each. Thus the area of Fig. 6.3(a) is equal to the area of Fig. 6.3(b). Now remove the four cardboard triangles to obtain the unshaded regions shown below:

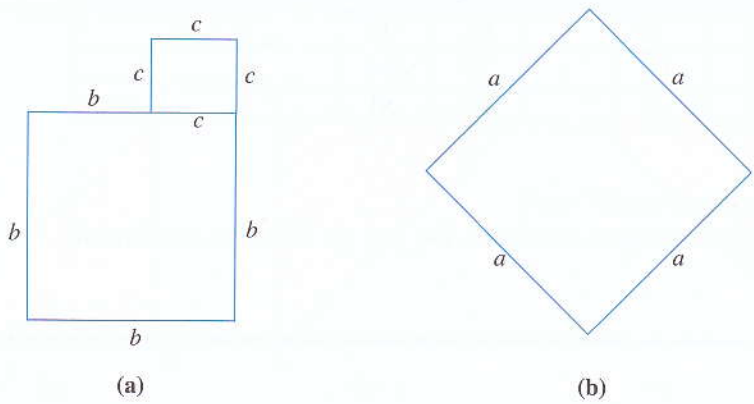


Fig. 6.4

Since the area of the cardboards removed from both figures are equal, the area of Fig. 6.4(a) must be equal to the area of Fig. 6.4(b), i.e., $a^2 = b^2 + c^2$.

We also have the following result.

In a triangle with sides a, b and c, if $a^2 = b^2 + c^2$, then the angle facing the side a is a right angle.

FOR YOUR Information

There are many interesting internet sites that provide interactive Java applets for proving Pythagoras' Theorem. Search the Internet to find out more.

Notes

A theorem is a fact that has been proven to be true.

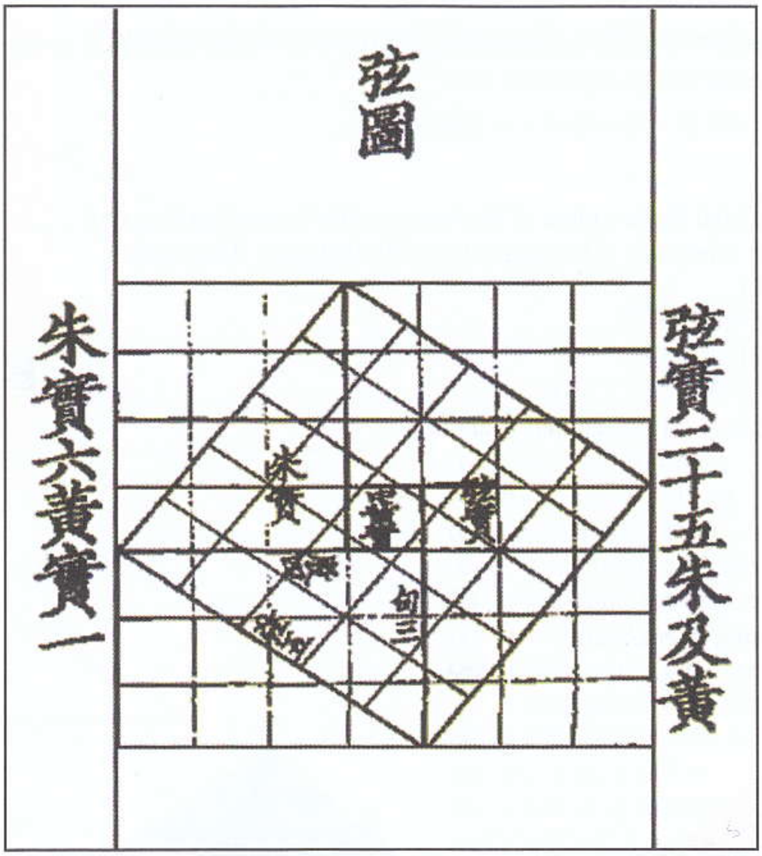
Journal Writing

Although Pythagoras was credited for discovering the Pythagoras' Theorem, the theorem was known thousands of years ago. The Babylonians were known to be familiar with the Pythagorean Triple (A set of 3 positive integers a, b and c which satisfy the equation $a^2 + b^2 = c^2$).

The Egyptian rope stretches made use of ropes knotted with a Pythagorean Triple for constructing a right angle during the time they constructed the great pyramids.

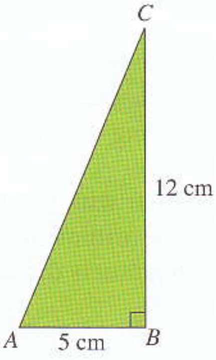
The Chinese were also familiar with this theorem. The diagram below known as "Hsuan-Thu" in Chinese illustrates the arithmetic-geometric methodology used in those times. The date of publication of the diagram is disputed to be ranging from 1200 BC to 100 AD.

Use the Internet to find out more about Pythogaras' Theorem, such as how it has evolved over the years and some well-known proofs of the theorem. Write a journal based on your findings and share your findings with your classmates.



Example 1

ABC is a triangle, where $A\hat{B}C = 90°$, AB = 5 cm and BC = 12 cm. Find the length of AC.



Solution

$AC^2 = AB^2 + BC^2$ (Pythagoras' Theorem)

$AC^2 = 5^2 + 12^2$

$= 25 + 144$

$= 169$

$AC = \sqrt{169}$ (since AC cannot be negative)

$\therefore$ $AC = 13$ cm

Example 2

Is the triangle whose sides are 15 cm, 36 cm and 39 cm a right-angled triangle?

Solution

A triangle is right-angled if the square of the longest side is equal to the sum of the squares of the other two sides (converse of Pythagoras' Theorem).

$39^2 = 1521$

$15^2 + 36^2 = 225 + 1296$

$= 1521$

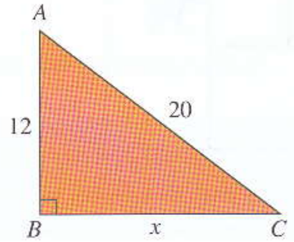
$\therefore$ $39^2 = 15^2 + 36^2$

$\therefore$ the given triangle is a right-angled triangle.

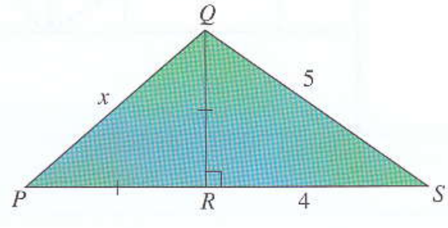
Example 3

Calculate the value of x in each case.

(a)



(b)



For Your Information

Pythagoras was a Greek philosopher and mathematician of Samos. In addition to Pythagoras' Theorem, he also discovered:

1. *the musical note produced by a vibrating string of a certain length is exactly one octave lower than the note produced by a string of the same material and half that length,*
2. *other notes in the musical scale can be produced by using certain fractions of the length of the string. For example, a string $\frac{4}{3}$ the length of a C-string produces the note G (one octave lower). The diagram below shows the various musical notes that can be produced for each given fraction of the length of a C-string.*

Solution

(a) $AB^2 + BC^2 = AC^2$

$12^2 + x^2 = 20^2$

$x^2 = 20^2 - 12^2$

$= 400 - 144$

$= 256$

$x = \sqrt{256}$

$\therefore \quad x = 16$

(b) $QR^2 + RS^2 = QS^2$

$QR^2 + 4^2 = 5^2$

$QR^2 = 5^2 - 4^2$

$= 25 - 16$

$= 9$

$QR = \sqrt{9}$

$\therefore \quad QR = 3$

$QR^2 + PR^2 = PQ^2$

$3^2 + 3^2 = x^2$

$x^2 = 18$

$\therefore \quad x = 4.24$ (correct to 3 sig. fig.)

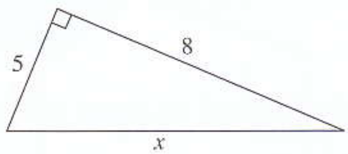
Pay Attention

Hypotenuse must always be by itself. So you cannot write: $x^2 = 12^2 + 20^2$.

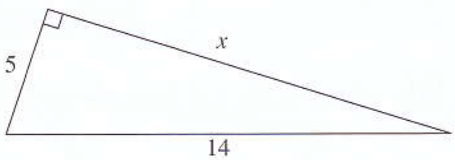
Exercise 6a

1. Calculate the value of x in each case.

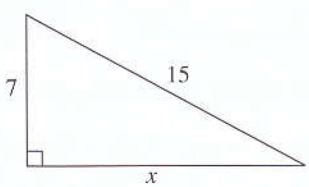
(a)



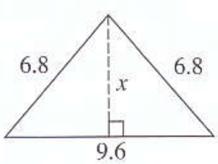
(b)



(c)

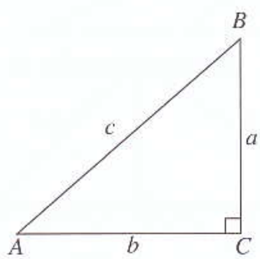


(d)



2. In the figure, $\hat{C} = 90°$. For each of the following, find the length of the unknown side.

(a) $a = 9$ cm, $b = 12$ cm

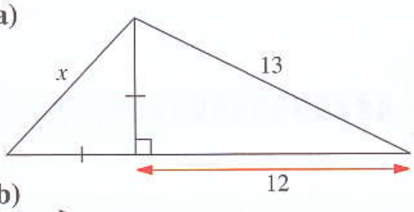


(b) $a = 15$ m, $b = 8$ m
(c) $b = 12$ m, $c = 13$ m
(d) $a = 7$ m, $c = 25$ m
(e) $a = 40$ cm, $c = 41$ cm
(f) $a = 21$ cm, $b = 20$ cm
(g) $a = 35$ cm, $c = 37$ cm
(h) $a = 11$ cm, $c = 61$ cm
(i) $a = 28$ cm, $c = 53$ cm
(j) $a = 33$ cm, $b = 56$ cm
(k) $a = 4$ m, $b = 7$ m
(l) $a = 6$ m, $c = 11$ m
(m) $c = 9$ cm, $b = 6$ cm
(n) $a = 6.7$ cm, $b = 5.5$ cm
(o) $c = 9.4$ cm, $a = 4.6$ cm
(p) $a = 14$ cm, $c = 19$ cm

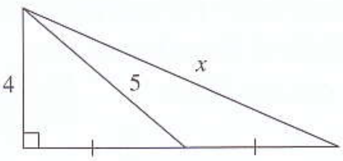
3. Which of the following could be the lengths of the sides of a right-angled triangle?
   (a) 8, 15, 17
   (b) 7, 25, 26
   (c) 9, 12, 15
   (d) 24, 45, 51
   (e) 24, 26, 10
   (f) 9, 17, 14
   (g) 0.15, 0.2, 0.25
   (h) $\frac{6}{13}, \frac{8}{13}, \frac{10}{13}$
   (i) 75, 23, 71
   (j) 0.8, 0.9, 1.2

4. Calculate the value of x in each case.

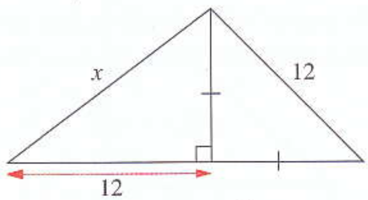
(a)



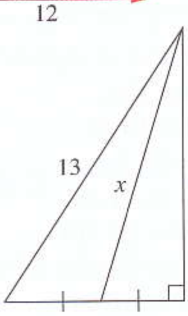
(b)



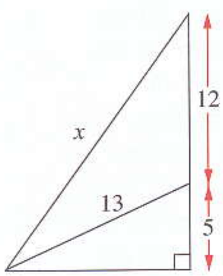
(c)



(d)

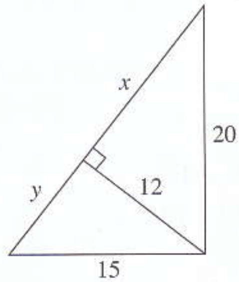


(e)

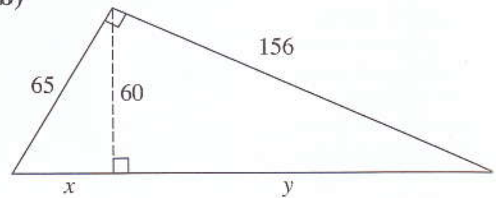


5. Find the values of x and y.

(a)

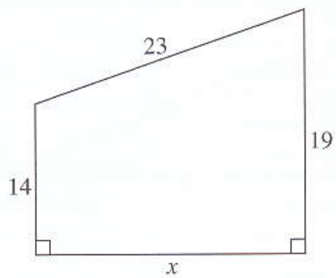


(b)

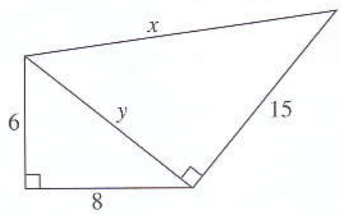


6. Calculate the value(s) of the unknown(s) in each of the following, giving your answer correct to 2 decimal places where necessary.

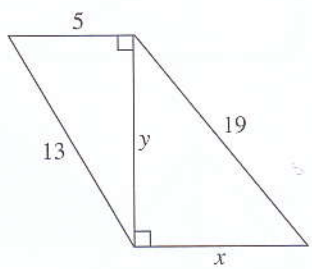
(a)



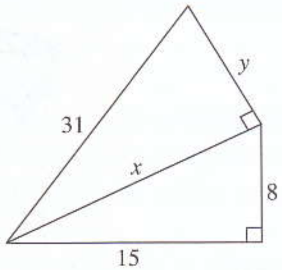
(b)



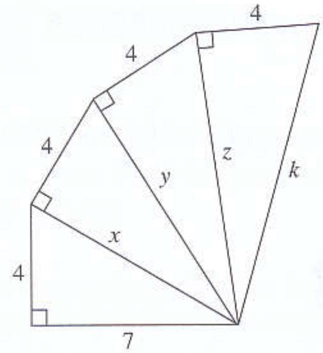
(c)



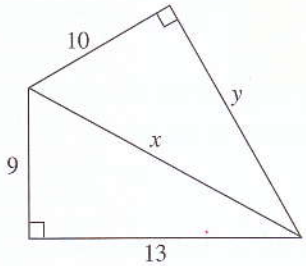
(d)



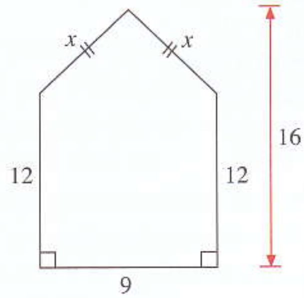
(g)



(e)



(g)

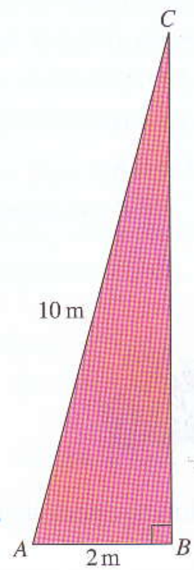


Applications of Pythagoras' Theorem

The following examples involve applying Pythagoras' theorem in solving problems.

Example 4

How high up the wall is a 10-m ladder if its foot is 2 m from the wall?



Solution

In the diagram, BC is the height that the ladder will reach.

$$AC^2 = AB^2 + BC^2$$
$$10^2 = 2^2 + BC^2$$
$$BC^2 = 10^2 - 2^2$$
$$= 100 - 4 = 96$$

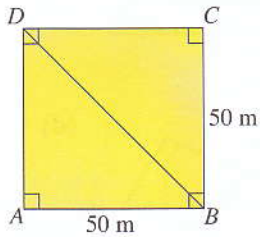
$\therefore$ $BC = \sqrt{96} = 9.80$ m (correct to 3 sig. fig.)

$\therefore$ the ladder will reach a height of 9.80 m.

Example 5

Each side of a square field ABCD is 50 m long. Find the length of the diagonal of the field.

Solution



$$
\begin{aligned}
BD^2 &= BC^2 + CD^2 \\
&= 50^2 + 50^2 \\
&= 2500 + 2500 \\
&= 5000 \text{ m}^2
\end{aligned}
$$

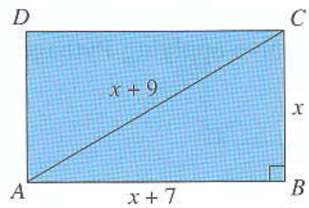
$\therefore \quad BD = \sqrt{5000}$

$= 70.7$ m (correct to 3 sig. fig.)

Example 6

The diagonal of a rectangular field ABCD is $(x + 9)$ m and the sides are $(x + 7)$ m and x m. Find the value of x.

Solution



$$
\begin{aligned}
AC^2 &= AB^2 + BC^2 \\
(x + 9)^2 &= (x + 7)^2 + x^2 \\
x^2 + 18x + 81 &= x^2 + 14x + 49 + x^2 \\
x^2 - 4x - 32 &= 0 \\
(x + 4)(x - 8) &= 0 \\
x = -4 &\text{ or } x = 8
\end{aligned}
$$

$\therefore \quad x = 8$ (We cannot have a negative length.)

Do it Yourself

How do you check that $x = 8$ is the correct answer?

For Your Information

Diophantus (200-284) posed this problem: If x, y, z and n are positive integers, under what condition does $x^n + y^n = z^n$ have a solution?

When $n = 2$, the equation $x^2 + y^2 = z^2$ has many solutions such as $x = 3$, $y = 4$ and $z = 5$ (Pythagorean Triples).

Fermat (1601–1665) wrote in the margin of his copy of the book containing this problem that he had found a proof to show that this problem does not have a solution if $n = 3$ but the margin was too small for him to write the proof. This became known as Fermat's last Theorem and it was the most famous marginal note because there were thousands of alleged proofs of his theorem but all of them were wrong!

Finally, Andrew Wiles proved Fermat's last Theorem in 1994: the proof was more than 200 pages long! But Wiles was over 40 years old at that time and so he was not eligible for the Field's Medal, which is the highest accolade for mathematics, equivalent in prestige and recognition to the Nobel Prizes, but given to outstanding mathematicians aged 40 years old or below.

Search the Internet to find out more about Fermat and Field's Medal.

Exercise 6b

1. A 5-m long ladder is placed against a wall with its foot 1.8 m from the wall. Find how far up the wall the ladder reaches.

2. A post 47 m high is supported by wires attached to its top and to a point on level ground, 18 m from the foot of the post. Find the length of each wire.

3. Find the length of the longest line segment that can be drawn on a rectangular board 3.07 m by 2.24 m.

4. $\triangle ABC$ is right-angled at B, and D is a point on BC. If $AD = 18$ cm, $BD = 9$ cm and $CD = 4$ cm, find AC.

5. The diagonals of a rhombus are of lengths 16 cm and 12 cm. Find the length of its sides.

6. The lengths of the sides of a triangle are x cm, $(x + 1)$ cm and $(x + 2)$ cm. Determine x so that this triangle is a right-angled triangle.

7. The sides of a rectangle are 24 cm and 15 cm. Calculate the length of its diagonal.

8. The sides of a rectangular swimming pool are 50 m and 30 m. What is the length between the opposite corners?

9. How far from the wall must you place a ladder of length 12 m, if the ladder is to touch a point 10 m above the ground?

10. A man runs diagonally across from one corner of a rectangular field 80 m by 60 m to the opposite corner in a straight line at a speed of 9 metres per second. Find the time taken for him to complete the run.

11. ABC is an equilateral triangle in which $BC = 2$ cm. Find the perpendicular distance from A to BC.

12. One diagonal of a rhombus is 24 cm. Find the length of the other diagonal if each side of the rhombus measures 13 cm.

13. $PQRS$ is a rectangle in which $PQ = 10$ cm and $PS = 6$ cm. T is a point on PQ such that RST is an isosceles triangle whose equal sides are RS and ST. Find RT.

14. $PQRS$ is a rectangle in which $PQ = 9$ cm and $PS = 6$ cm. T is a point on PQ such that $PT = 7$ cm and RV is the perpendicular from R to ST. Calculate ST and RV.

15. In a parallelogram $ABCD$, the diagonal AC is at right angle to AB. If $AB = 12$ cm and $BC = 13$ cm, find the area of the parallelogram.

16. A rectangle is 7 cm wide, and the length of each diagonal is 25 cm. Find the length of the rectangle and its area.

*17. Three integers, a, b, and c, with $a < b < c$, are said to form a Pythagorean Triple if $c^2 = a^2 + b^2$. For example, 3, 4, 5 form a Pythagorean Triple because

$$3^2 + 4^2 = 9 + 16 = 25 = 5^2$$

(a) In the same way, show that 6, 8 and 10 form a Pythagorean Triple.

(b) Form a Pythagorean Triple
  (i) in which the first two integers are 12 and 16,
  (ii) in which the last integer is 25.

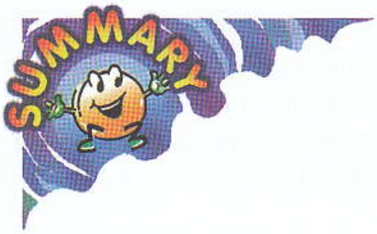
(c) **(i)** Simplify $(3n)^2 + (4n)^2$.

(ii) Hence form a Pythagorean Triple in which the last integer is 35.

(d) Consider $1 + 2n + n^2 = (1 + n)^2$.
If $1 + 2n$ is a perfect square, say, $1 + 2n = k^2$, we have $k^2 + n^2 = (1 + n)^2$. Then k, n, $1 + n$ is a Pythagorean Triple. For example, when $n = 12$, $1 + 2n = 25 = 5^2$, and then 5, 12, 13 form a Pythagorean Triple.

(i) In the same way, find the corresponding Pythagorean Triple when $n = 24$.
Can we get a Pythagorean Triple in the same way when $1 + 2n = 42$? Why?

(ii) Form a Pythagorean Triple such that the first integer is 9 and the other two integers differ by 1.

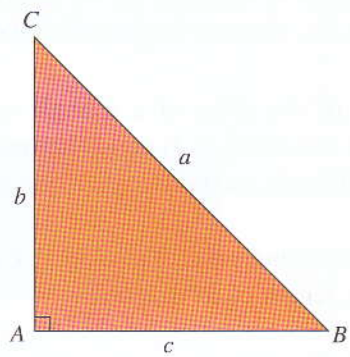


1. Pythagoras' Theorem:

For a right-angled triangle ABC,

$$BC^2 = AC^2 + AB^2$$

i.e., $$a^2 = b^2 + c^2$$

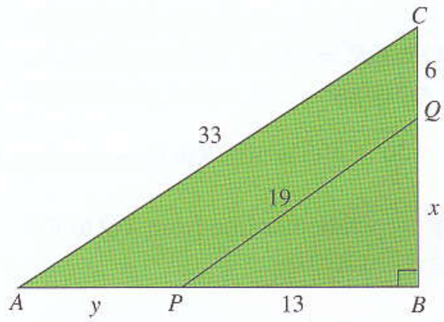


2. The converse of the Pythagoras' Theorem states that if $a^2 = b^2 + c^2$, then the triangle with sides a, b and c is a right-angled triangle, with the angle opposite the side a being a right angle.

Review Examples 6

Example 1

In the diagram $A\hat{B}C = 90°$,
$AC = 33$ cm, $PB = 13$ cm,
$CQ = 6$ cm, $PQ = 19$ cm,
$BQ = x$ cm and $AP = y$ cm.
Calculate the value of x and of y.



Solution

In ΔPBQ, $PQ^2 = PB^2 + BQ^2$

$$19^2 = 13^2 + x^2$$

$$\therefore \quad x^2 = 19^2 - 13^2 = 192$$

$$x = \sqrt{192} = 13.856$$

$$\therefore \quad x = 13.9 \text{ cm (correct to 3 sig. fig.)}$$

In ΔABC, $AC^2 = AB^2 + BC^2$

$$33^2 = (y + 13)^2 + (6 + 13.856)^2$$

$$(y + 13)^2 = 33^2 - 19.856^2 = 694.74$$

$$y + 13 = \sqrt{694.74} = 26.358$$

$$y = 13.4 \text{ cm (correct to 3 sig. fig.)}$$

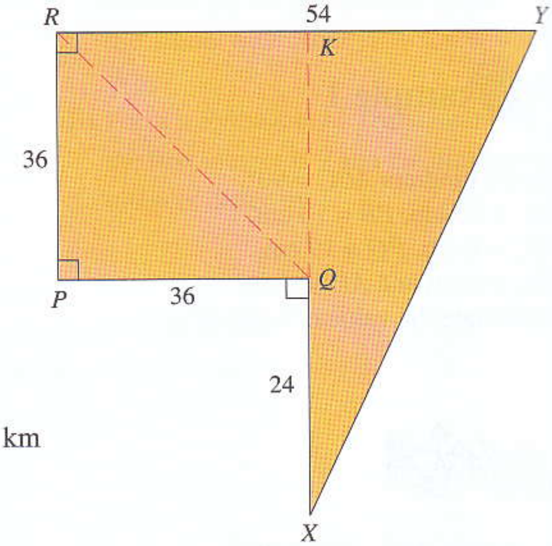
Example 2

Two boats A and B leave a port P at the same time. Boat A travels due east for 3 hours to reach a buoy Q and then travels due south for another 2 hours to reach a port X. Boat B travels due north for 2 hours to reach a buoy R and then travels due east for 3 hours to reach a port Y. If the average speed of boats A and B are 12 km/h and 18 km/h respectively, calculate the distance between

(a) the buoys Q and R,

(b) the ports X and Y.

Solution



Now $PQ = 3 \times 12 = 36$ km
$QX = 2 \times 12 = 24$ km
$PR = 2 \times 18 = 36$ km
$RY = 3 \times 18 = 54$ km

(a) In ΔPQR, $QR^2 = PQ^2 + PR^2$
$QR^2 = 36^2 + 36^2$
$QR = \sqrt{36^2 + 36^2} = 50.9$ km (correct to 3 sig. fig.)

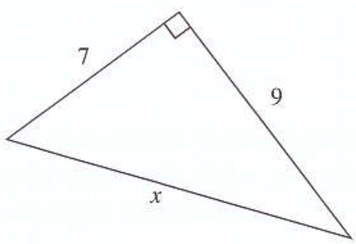
(b) Drop a perpendicular line from XQ to RY cutting RY at K.

Now $XK = 36 + 24 = 60$ km and $KY = 54 - 36 = 18$ km
$XY^2 = XK^2 + KY^2$
$XY = \sqrt{60^2 + 18^2}$
$= 62.6$ km (correct to 3 sig. fig.)

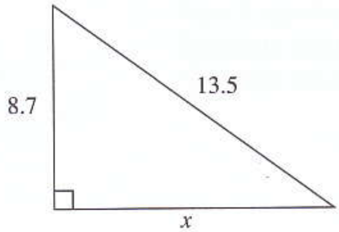
Review Questions 6

1. Calculate the value of x in each of the following:

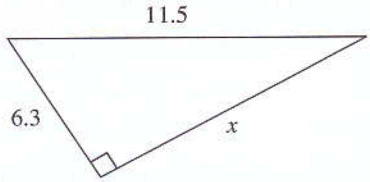
(a)



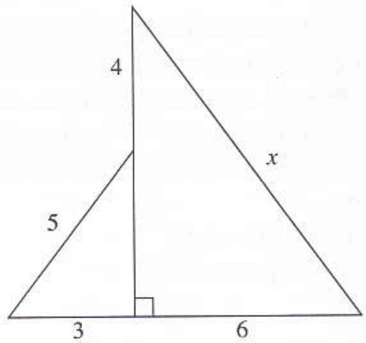
(b)



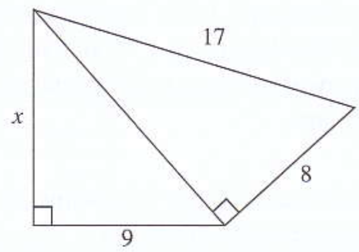
(c)



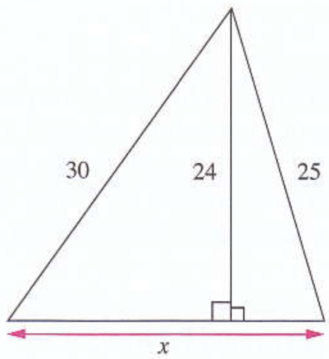
(d)



(e)



(f)



2. The sides of a rectangle are 14 cm and 25 cm. Calculate the length of the diagonal.

3. The diagonal of a square is 42.5 cm. Calculate the perimeter and area of the square.

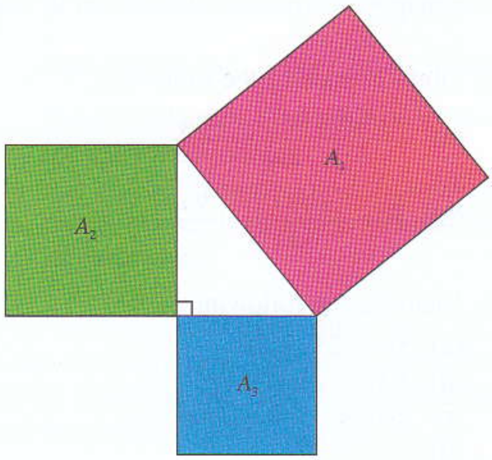
4. The length of the sides of a rhombus is 52 cm. One of its diagonals is 48 cm. Find the length of the other diagonal and the area of the rhombus.

5. Find the distance from the mid-point of one side of a square of length 10 cm to either end of the opposite side.

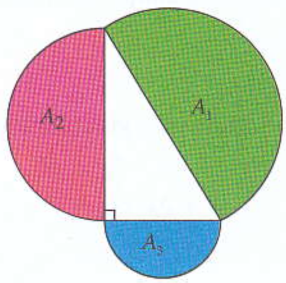
6. PQR is an isosceles triangle such that $QP = QR = 13$ cm and the altitude $PS = 12$ cm. Find the length of PR.

*7. Given a triangle ABC where the value of $BC^2 = 370$ units, $AC^2 = 74$ and $AB^2 = 116$, calculate the area of the triangle.
[Hint: $370 = 9^2 + 17^2$, $74 = 5^2 + 7^2$ and $116 = 4^2 + 10^2$]

*8. Pythagoras' Theorem is usually stated in terms of the sides of a right-angled triangle: $a^2 + b^2 = c^2$. But it can also be stated in terms of areas of squares: $A_1 = A_2 + A_3$ (see diagram below).



What if the areas do not refer to the areas of squares but areas of semicircles (see diagram below)?
Does the relationship still work? Is A_1 still equal to $A_2 + A_3$? Why or why not?



Revision Exercise II No. 1

1. A semi-circular flower bed has a perimeter of 72 cm. Calculate the area of the flower bed, taking π to be 3.142.

2. Solve the following equations:
   (a) $5(x^2 - 9) = 3(x - 3)^2$
   (b) $(2 - x)(4 + x) + x^2 = 14$

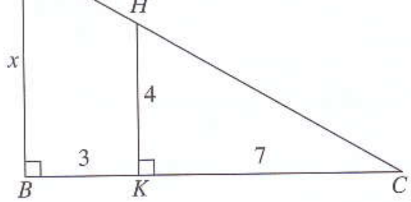
3. What is the Representative Fraction (R.F.) of a map which has a scale of 1 cm to 2.5 km?

4. Simplify each of the following:
   (a) $\dfrac{4p^2q^2 - 8p^3q - 14pq^3}{2pq}$
   (b) $\dfrac{6a^3b^4 + 12a^4b^2 - 9a^2b^3}{-3a^2b^2}$

5. Factorise the following:
   (a) $5x^2 - 25x + 20$
   (b) $10a^2 - 21ab + 9b^2$
   (c) $28a^2 + 11ab - 24b^2$
   *(d) $a^2b^2 - 25$

6. (a) If $5a = \dfrac{3x - 4}{2y - 3x}$, make x the subject of the formula and use your result to obtain the value of x when $a = 1$ and $y = 5$.
   (b) Make a the subject of the formula
   $$3x = \frac{a + b}{2a - b}.$$
   (c) Make x the subject of the formula
   $$y = \frac{nx}{k - hx}.$$

7. Solve the following simultaneous equations:
   $$5x - 2y = 16,$$
   $$x + 3y + 7 = 0.$$

8. 

   In the figure, $AB = x$ cm, $HK = 4$ cm, $BK = 3$ cm and $KC = 7$ cm. Find the value of x and the length of CH and of AH, giving your answer correct to 2 decimal places where necessary.

9. Express each of the following as a fraction with a single denominator.
   (a) $\dfrac{5(a + b)}{3} + \dfrac{a - b}{5}$
   (b) $\dfrac{1}{2a} - \dfrac{2}{7a} + \dfrac{1}{3}$

10. If the numerator of a fraction is increased by 1, the value of the fraction becomes 1. If the numerator is increased by 2 and the denominator is decreased by 3, the value of the fraction becomes 2. What is the fraction?

Revision Exercise II No. 2

1. How long will it take a car to cross a bridge 900 m long if it is travelling at 54 km/h?

2. Write down the expansion of
   (a) $(a + b)(a^2 - ab + b^2)$,
   (b) $(x - y)(x^2 + xy + y^2)$.

3. Copy and complete the following sequence of numbers:
   (a) 4, 5, 8, 13, 20, ______, ______.
   (b) 3, 4, 8, 17, 33, ______, ______.
   (c) $\frac{7}{8}, \frac{2}{3}, \frac{1}{2}, \frac{4}{11}, \frac{1}{4}$, ______, ______.

4. Solve the following equations:
   (a) $\dfrac{x}{2} + \dfrac{2x}{5} = \dfrac{x}{15} + 3$
   (b) $\dfrac{2x - 5}{6} - \dfrac{x - 2}{4} = \dfrac{3}{8}$
   (c) $x(2 - 3x) + 1 = 0$
   (d) $5x(x + 1) - 18 = 2x(2x - 1)$

5. If $x + y = 2$ and $x^2 + y^2 = 8$, find the value of
   (a) xy;
   (b) $x - y$.

6. (a) Make x the subject of the formula
$\frac{1}{a} = \frac{2}{b} - \frac{3}{x}$.
(b) Make a the subject of the formula
$F = \frac{2ab - 4c}{3a - 5}$

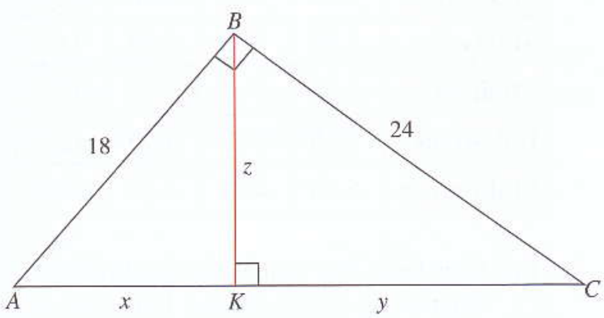
7. Given that y is directly proportional to the cube of x, copy and complete the following table, and then express y in terms of x.

x	1	2	
y	3		192

8. (a) Solve the simultaneous equations
$2(x - 1) + 3(y + 1) = 4,$
$5(x - 2) + 2(y + 2) = 7.$
(b) A man paid \$101 for 2 pairs of trousers and 3 shirts. He found that 7 pairs of trousers and 9 shirts would cost \$331. Find the cost of
(i) a pair of trousers,
(ii) a shirt.

9. Factorise the following:
(a) $4x^2 + 24x + 35$
(b) $5x^2 - 2x - 3$
(c) $6x^2 - 23x + 21$

10. In the diagram, $A\hat{B}C = B\hat{K}C = 90°$, $AB = 18$ cm, $BC = 24$ cm, $AK = x$ cm, $BK = z$ cm and $KC = y$ cm. Calculate the value of x, of y and of z.



Revision Exercise II No. 3

1. (a) Given that $S = \frac{a}{1 - r}$, express r in terms of S and a. Find r when $S = 11\frac{1}{9}$ and $a = 10$.
(b) Given that $\frac{1}{a} = \frac{1}{2b} + \frac{1}{3c}$, express c in terms of a and b. Hence, or otherwise, evaluate c when $a = 2$ and $b = 5$.

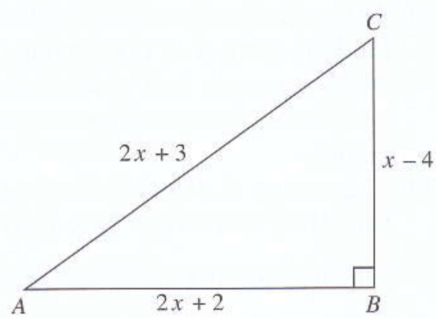
2. (a) A car travels 75 km in $1\frac{1}{4}$ hours and then travels for $\frac{1}{2}$ hour at an average speed of 80 km/h. Find the average speed for the whole journey.
(b) One side of a rectangle is 2 cm longer than the other. If its area is 195 cm^2, find its perimeter.

3. Two felt pens and three rulers cost \$3.30 while five felt pens and seven rulers cost \$8.10. Calculate the cost of three felt pens and four rulers.

4. Solve the following equations:
(a) $\frac{x}{3} - \frac{x}{4} = \frac{3}{8}$
(b) $\frac{2}{3a} + \frac{1}{4a} = \frac{5}{6}$

5. A boy paid \$1.60 for 4 exercise books and 2 ball-point pens. If a ball-point pen costs 5¢ more than an exercise book, find the cost of each ball-point pen and each exercise book.

6.



The figure above shows a right-angled triangle ABC where $AB = (2x + 2)$ cm, $BC = (x - 4)$ cm and $AC = (2x + 3)$ cm. Form an equation in x and hence, find the length of BC, AB and AC.

7. When a satellite orbits the earth, the force F attracting it towards the earth is inversely proportional to r^2 where r km is the distance of the satellite from the centre of the earth. Given that $F = 60$ units when $r = 30$, calculate the value of
   (a) r when $F = 240$ units,
   (b) F when $r = 45$.

8. Factorise the following expressions:
   (a) $x^2 - 38x + 345$
   (b) $x(x + 2) + x(x - 5) - 5$
   (c) $9(x^2 + y^2) - 25y^2$

9. Solve the following equations:
   (a) $2(x^2 - 9) = (x + 3)(x - 6)$
   (b) $x(x^2 - x - 12) = (x + 2)(x + 3)^2$

10. Simplify the following expressions:
   (a) $(x + 2)(x - 5) - 3x(x - 4) + 7x(x + 3)$
   (b) $\dfrac{1}{x-1} + \dfrac{1}{x+1}$
   ∗**(c)** $\dfrac{2x+3}{x^2+1} - \dfrac{x}{x+1} - \dfrac{5}{x-1}$

Revision Exercise II No. 4

1. Solve the following equations:
   (a) $\dfrac{4x}{9} - \dfrac{x}{6} = \dfrac{1}{5}$ **(b)** $\dfrac{7x-3}{5x} + \dfrac{1}{x} = 3$

2. **(a)** A bicycle wheel has a diameter of 29 cm. Find the number of revolutions it makes when it travels half a kilometre. Give your answer correct to the nearest whole number.
   (b) The energy, E, of an object of mass m kg travelling at a height of h metres with velocity v m/s is $E = \left(\dfrac{mv^2}{2} + mgh\right)$ joules. Make m the subject of the formula.

3. The length of a diagonal of a rectangular field is 36.8 m and one of its sides is 23.8 m. Find the area and perimeter of the rectangle.

4. The period P seconds of the oscillation of a simple pendulum is directly proportional to the square root of its length l cm. When the length is 49 cm, the period is 3.8 seconds. Find
   (a) the period when the length is 121 cm.
   (b) the length when the period is 3 seconds.

5. Factorise
   (a) $(x + 2)^2 + x^2 + 6x + 8$,
   (b) $(x + y)^2 + 5(x + y) + 6$,
   ∗**(c)** $a^4 - b^4$.

6. Solve the simultaneous equations

$$\frac{1}{7}(x + 2) + y = 4,$$
$$\frac{1}{4}(x - 1) + 2y = 7.$$

7. Make x the subject of the formulae:
   (a) $\dfrac{x}{a} - b = \dfrac{x}{c}$, **(b)** $ax + b = cx - d$.

8. If 1 is added to both the numerator and the denominator of a fraction, its value becomes $\dfrac{3}{4}$. If 3 is subtracted from both the numerator and the denominator, its value becomes $\dfrac{1}{2}$. Find the fraction.

9. The table below shows the indices to measure the quality of the teaching of science in five Asian countries for the years 1996 to 1999, 1 being the worst and 10 being the best quality.

	1996	1997	1998	1999
Singapore	8.08	8.15	8.16	8.73
Malaysia	5.89	7.00	6.85	7.02
Thailand	3.47	4.15	3.72	4.72
Indonesia	4.90	4.53	4.40	4.57
Philippines	5.18	4.34	4.77	4.91

   (a) Construct line graphs for the indices of Singapore, Malaysia and Indonesia for the period 1996 to 1999 in a single diagram.

(b) Calculate the percentage increase in the index for Malaysia from 1996 to 1997.

(c) Calculate the percentage change in the index for Singapore from 1997 to 1998 and from 1998 to 1999. Can you suggest a reason for the small percentage increase from 1997 to 1998 for Singapore?

(d) Suggest a reason for the drop in the indices for Malaysia, Thailand and Indonesia from 1997 to 1998.

Revision Exercise II No. 5

1. **(a)** A boy used 3.214 for π instead of 3.142 in calculating the area of a circle of radius 7.6 cm. Find the percentage error giving your answer correct to 3 significant figures.

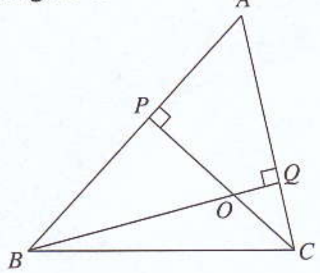
(b) The resistance R ohms of a wire of a constant length is inversely proportional to the square of its diameter d cm. If $R = 40$ when $d = 0.8$, find the value of R when $d = 0.6$.

2. **(a)** The ratio of the weights of two boys is 7 : 8. If the heavier boy weighs 48 kg, what is the weight of the lighter boy?

(b) A group of children shared \$$x$ among themselves. Each of them got \$7 and there were \$4 left. If there were \$5 more, each would get \$8. Find x and the number of children.

3. Given that $A = k(R^2 - r^2)$, express k in terms of A, R and r. Find the value of k if $A = 50$, $R = 5$ and $r = 3$.

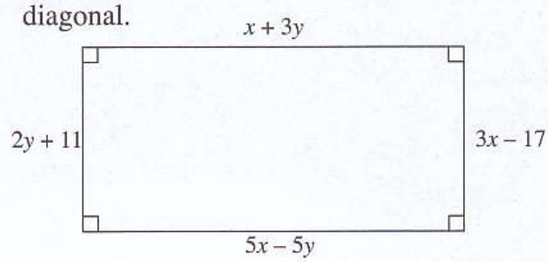
4. In the figure below, $A\hat{P}C = A\hat{Q}B = 90°$. $OP = 20$ cm, $OB = 38$ cm and $OC = 12$ cm. Calculate the lengths of PB and OQ.



5. Solve the simultaneous equations

$$3x + 2y = -2,$$
$$\frac{3x+2}{2} + \frac{y-2}{3} = 0.$$

6. In the given figure, the lengths are in cm. Find the area of the rectangle and the length of its diagonal.



7. **(a)** If $v = u + at$, find t in terms of u, v and a. Find the value of t when $v = 12$, $u = 2$ and $a = 4$.

(b) The product of two consecutive positive integers is 240. Find the two numbers.

8. Simplify the following expressions:

(a) $\frac{2}{x+2} - \frac{3}{x-3}$

(b) $\frac{x-5}{x+6} - \frac{x+5}{x-6}$

(c) $\frac{1}{2x} - \frac{3}{x+6}$

9. Make x the subject of the formulae:

(a) $a = bx + x$

(b) $a = \frac{ax+b}{cx-b}$

10. **(a)** Factorise

*(i) $x^4 + x^2 - 2$,

*(ii) $x^4 - 13x^2 + 36$.

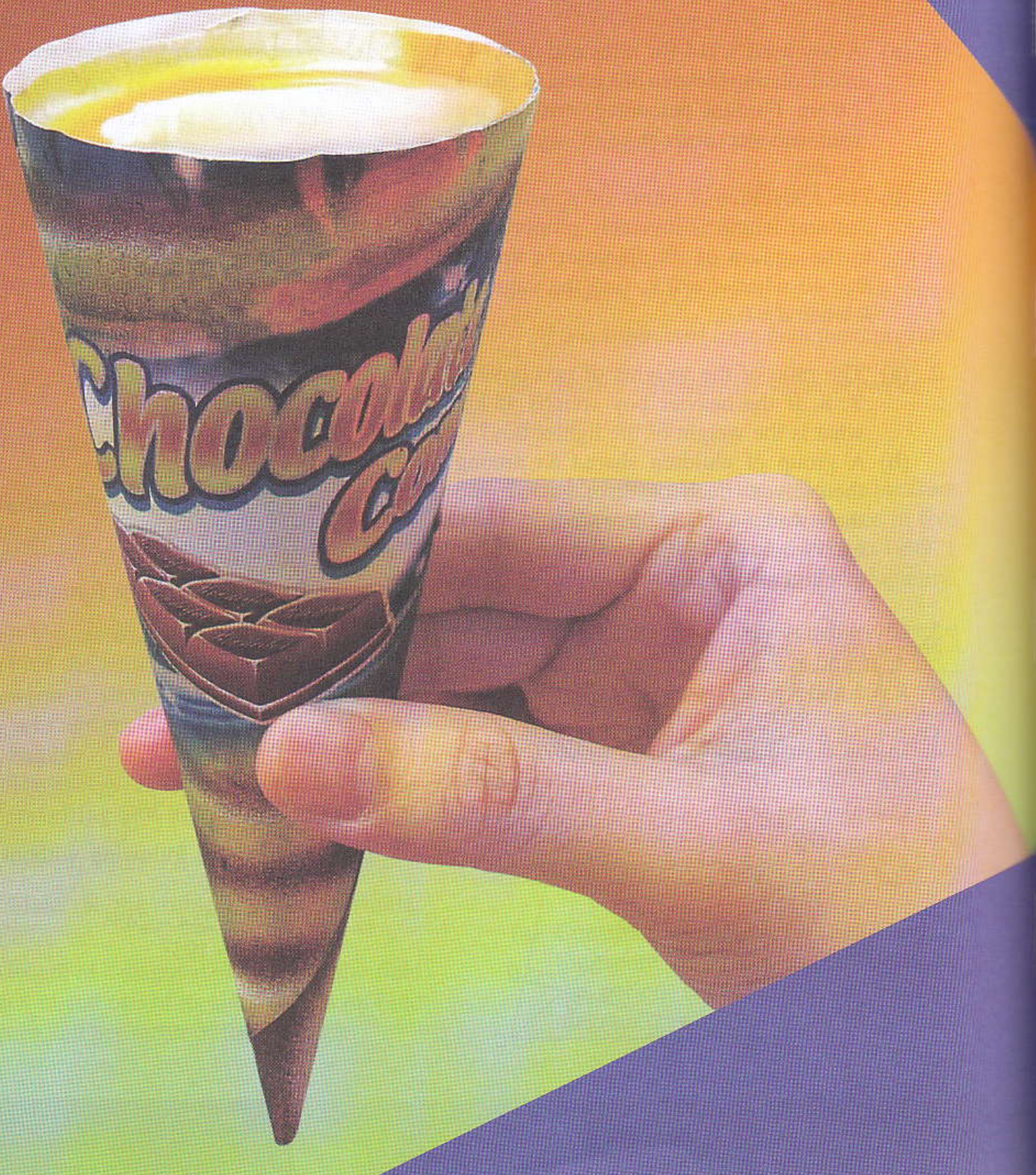
(b) Evaluate the following without using a calculator:

(i) $243 \times 244 - 243^2$

(ii) $196^2 - 195^2$

In this chapter, you will learn how to

- *draw sketches of pyramids, cones and spheres;*
- *draw the nets of pyramids and cones;*
- *find the volume and surface area of pyramids, cones and spheres;*
- *solve problems involving volume and surface area of composite solids made up of a combination of pyramids, cones, spheres, prisms and cylinders.*



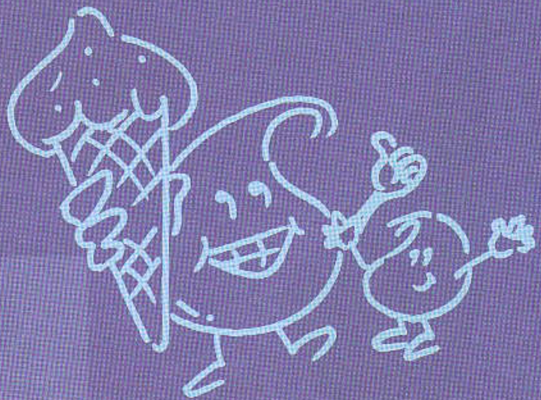
CHAPTER

7

Volume and Surface Area

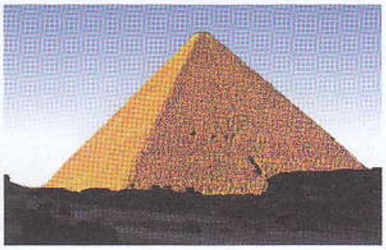
Introduction

The photo shows an ice-cream cone. How does the manufacturer determine the volume of ice-cream needed to fill up the cone completely?



Pyramids

The photo in Fig. 7.1(a) shows the Great Pyramid built by the Egyptians in Giza, Egypt, around the year 2560BC. The photo in Fig 7.1 (b) shows two dice with 4 triangular sides each. These are examples of pyramids. What are some common features of these pyramids?



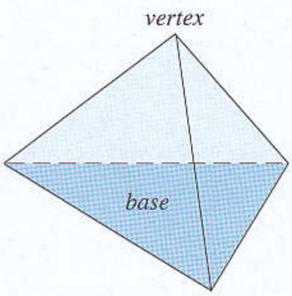
(a) The Great Pyramid

(b) Tetrahedral Dice with Four Faces

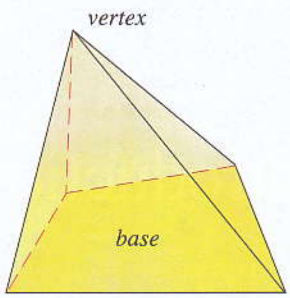
Fig. 7.1 Examples of Pyramids

> **FOR YOUR Information**
>
> *When you toss a tetrahedral die with four sides numbered 1, 2, 3 and 4, your score is the number on the side facing down. But it is difficult to read this number. So the dice are numbered as shown in Fig. 7.1(b). The score is the number closest to the base. The left die shows a score of 2 and the right die shows a score of 3.*

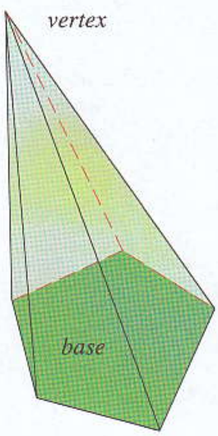
A **pyramid** is a solid with a **polygonal base** and a **vertex**. All the other faces are triangles which meet at this vertex. Fig. 7.2 shows some examples of pyramids with different bases. The pyramid is named after its polygonal base.



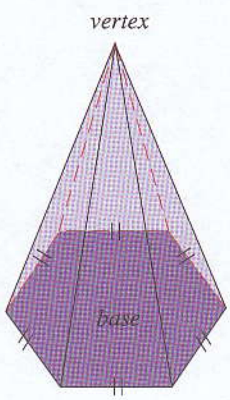
(a) Triangle-based Pyramid (or Tetrahedron)



(b) Quadrilateral-based Pyramid



(c) Pentagon-based Pyramid



(d) Hexagon-based Pyramid

Fig. 7.2 Pyramids with Different Bases

Fig. 7.3 shows three pyramids. For Pyramid (a) and Pyramid (b), the vertex is vertically above the centre of the base. This type of pyramid is known as a right pyramid. Why is Pyramid (c) not a right pyramid?

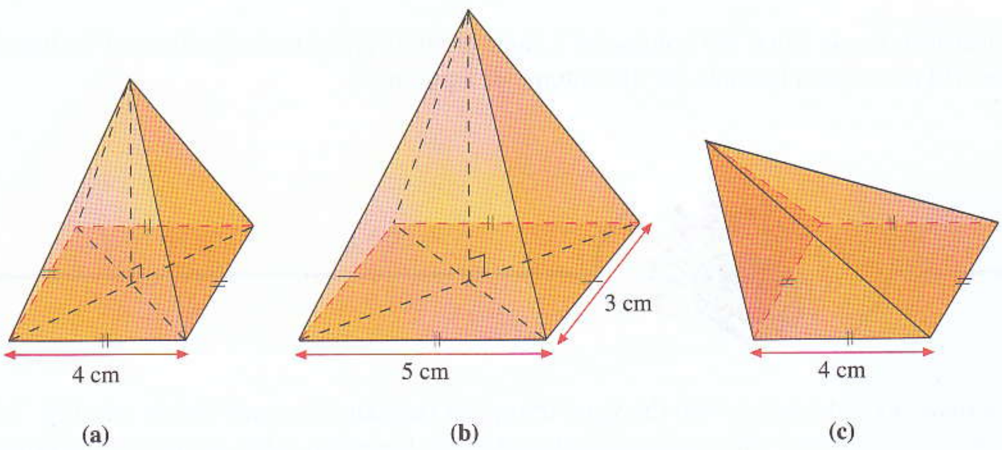


Fig. 7.3 Which are right pyramids?

Non-right pyramids are called oblique pyramids.

The perpendicular height (or height) of a pyramid is the perpendicular distance from the vertex to the base of the pyramid (see Fig. 7.4). A slant height of a pyramid is the distance from the vertex to the mid-point of an edge of the base. The edges joining the vertex to the corners of the base of the pyramid are called the slant edges.

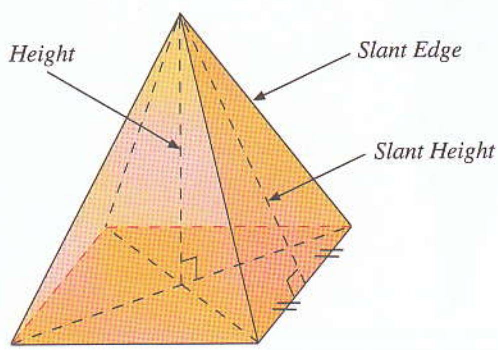


Fig. 7.4 Height, Slant Height and Slant Edge.

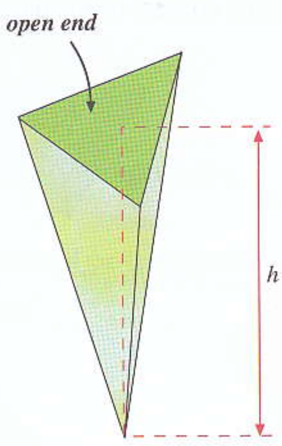
In this chapter, we will study only right pyramids. For simplicity, we will use the term "pyramid" to mean right pyramid, unless otherwise specified. We will also use the term "height" to mean the perpendicular height of the pyramid.

Volume of a Pyramid

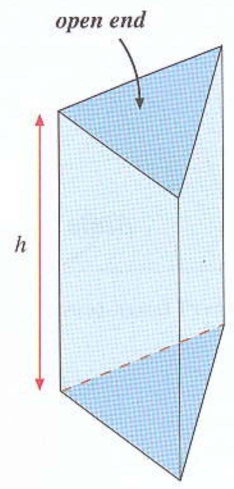
We have learnt in Book 1 that the volume of a right prism is equal to the product of its base area and height. We will now find a formula for the volume of a pyramid.

Explorations

Consider a pyramid and a prism with the same triangular base and the same height (see Fig. 7.5). If we are to fill the whole pyramid with sand and then pour it into the prism, what fraction of the prism will be filled? If we repeat the process, how many times will it take to fill the prism completely?



(a) Triangle-based Pyramid



(b) Triangular Prism

Fig. 7.5 Pyramid and Prism

Fig. 7.6 shows the net of a triangle-based pyramid that you can photocopy and paste on a cardboard before cutting it out and folding along the dotted lines to obtain an open pyramid as shown in Fig. 7.5(a). The tabs are for gluing purposes. An open pyramid is a pyramid without the base.

The tabs are not part of the net.

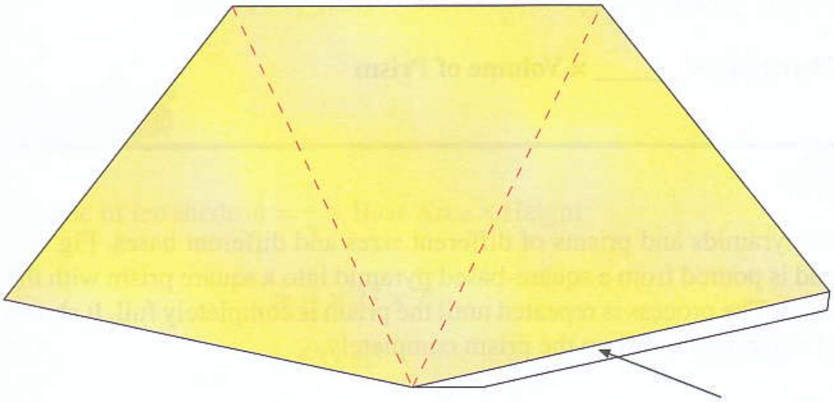


Fig. 7.6 Net of Triangle-based Pyramid

Using the cardboard is necessary. If you just use paper, the pyramid will be distorted under the weight of the sand and the result will be very inaccurate.

Fig. 7.7 shows the net of a triangular prism that you can photocopy and paste on a cardboard before cutting it out and folding along the dotted lines to obtain an open prism as shown in Fig. 7.5(b). Notice that both the pyramid in Fig. 7.6 and the prism in Fig. 7.7 have the same triangular base and the same height.

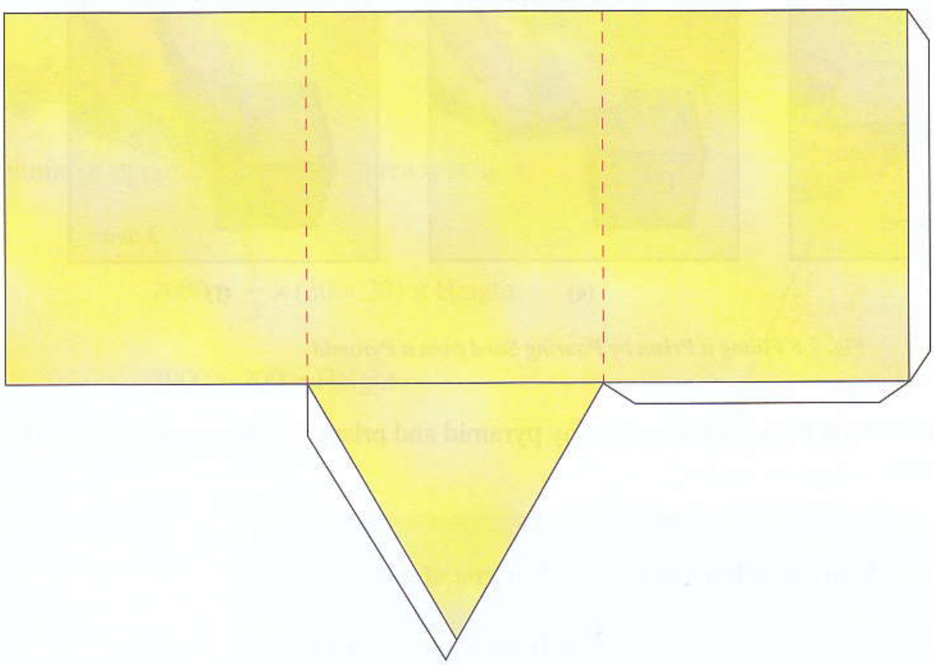


Fig. 7.7 Net of Triangular Prism

Fill the whole pyramid with sand such that the sand is level. Pour the sand into the prism. How many times must you do this before the prism is completely filled?

Do you get the same result as your classmates?

What conclusion can you get about the volume of a pyramid and the volume of a prism?

Volume of Pyramid = ______ × Volume of Prism

You can repeat the experiment for pyramids and prisms of different sizes and different bases. Fig. 7.8 shows a series of photos where sand is poured from a square-based pyramid into a square prism with the same square base and the same height. The process is repeated until the prism is completely full. It shows that it takes 3 times the volume of a pyramid to fill up the prism completely.

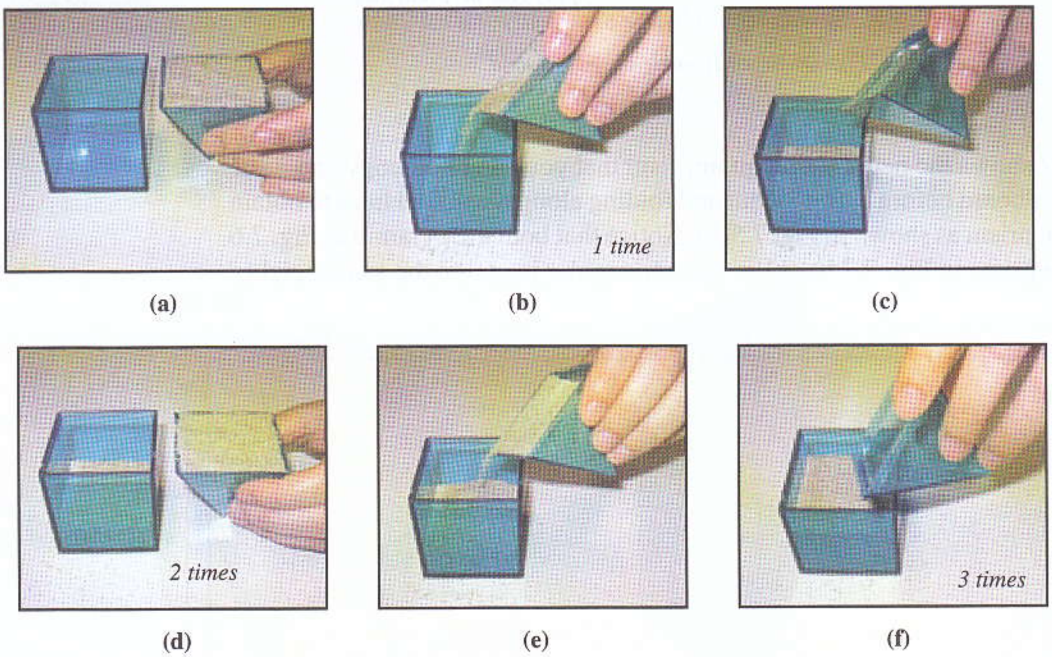


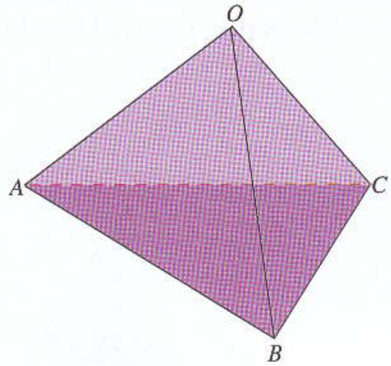
Fig. 7.8 Filling a Prism by Pouring Sand from a Pyramid

This suggests that the following formula is true for any pyramid and prism with the same polygonal base and the same height:

$$\textbf{Volume of Pyramid} = \frac{1}{3} \times \textbf{Volume of Prism}$$
$$= \frac{1}{3} \times \textbf{Base Area} \times \textbf{Height}$$

Example 1

OABC is a tetrahedron (or triangle-based pyramid). Given that the area of the base is 27 cm² and the height is 7 cm, find the volume of the tetrahedron.



Solution

$$\text{Volume of tetrahedron} = \frac{1}{3} \times \text{Base Area} \times \text{Height}$$

$$= \frac{1}{3} \times 27 \times 7$$

$$= 63 \text{ cm}^3$$

Example 2

VPQRS is a rectangle-based pyramid where PQ = 30 cm and QR = 20 cm. Given that the volume of the pyramid is 2000 cm³, find its height.

Solution

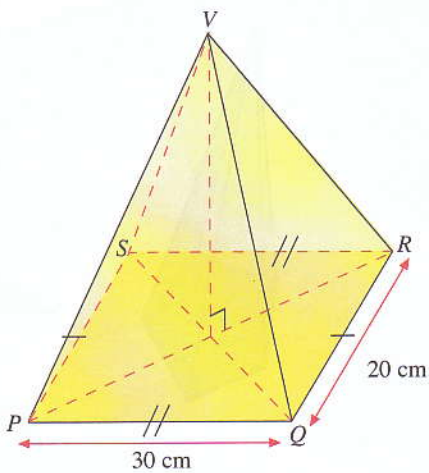
$$\text{Volume of pyramid} = \frac{1}{3} \times \text{Base Area} \times \text{Height}$$

$$2000 = \frac{1}{3} \times (30 \times 20) \times \text{Height}$$

$$2000 = 200 \times \text{Height}$$

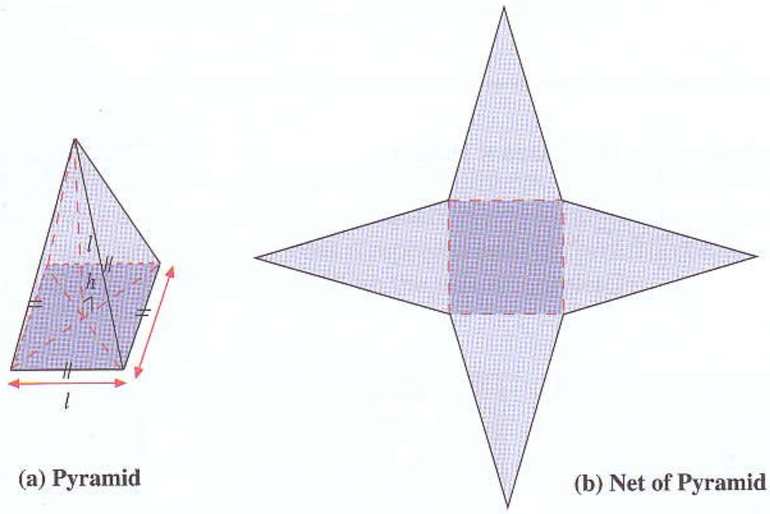
$$\text{Height} = \frac{2000}{200}$$

$$= 10 \text{ cm}$$



Total Surface Area of Pyramid

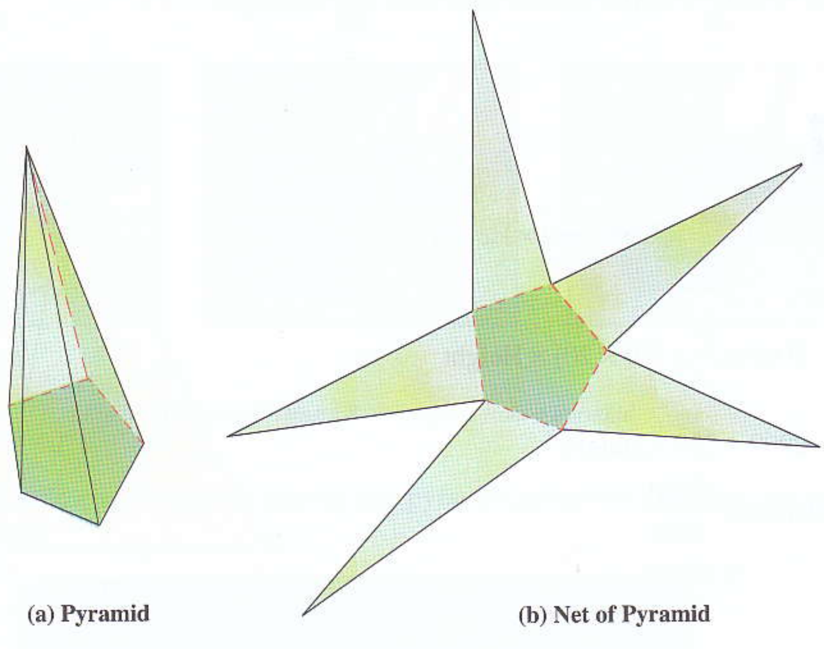
Fig. 7.9 shows (a) a pyramid with a square base and (b) its net. If you fold the net along the dotted lines, you will obtain the pyramid. How do you find the total surface area of the pyramid?



(a) Pyramid (b) Net of Pyramid

Fig. 7.9 Net of Pyramid

Fig. 7.10 shows a pyramid with a pentagonal base and its net. How do you find the total surface area of the pyramid?



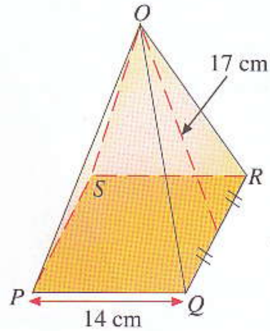
(a) Pyramid (b) Net of Pyramid

Fig. 7.10 Net of Pyramid

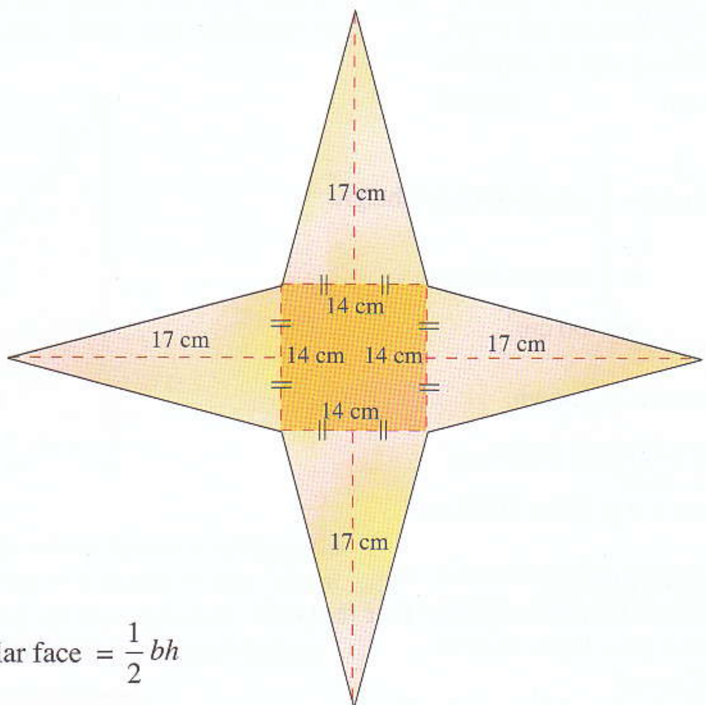
Therefore, **Total Surface Area of Pyramid = Total Area of All its Faces**

Example 3

OPQRS is a right pyramid whose base is a square of sides 14 cm each. Given that the slant height of the pyramid is 17 cm, draw the net of the pyramid and find its total surface area.



Solution



Area of each triangular face $= \frac{1}{2}bh$

$= \frac{1}{2} \times 14 \times 17$

$= 119 \text{ cm}^2$

Area of square base $= 14 \times 14$

$= 196 \text{ cm}^2$

$\therefore$ total surface area of pyramid $= 4 \times$ area of triangular face $+$ area of square base

$= 4 \times 119 + 196$

$= 672 \text{ cm}^2$

Example 4

In Fig. 7.11, OPQRS is a pyramid whose base is a square of sides 30 cm each. The slant height OM =17 cm. Find

(a) the height of the pyramid ON,
(b) the volume of the pyramid,
(c) the total surface area of the pyramid.

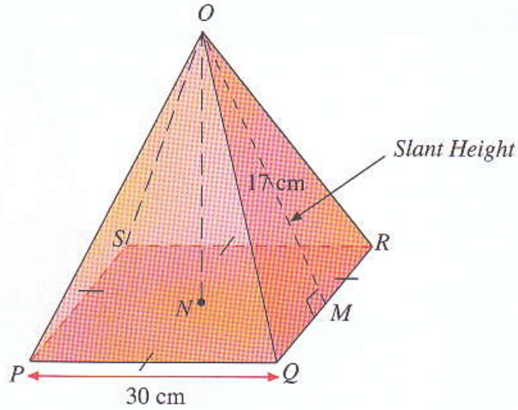


Fig. 7.11

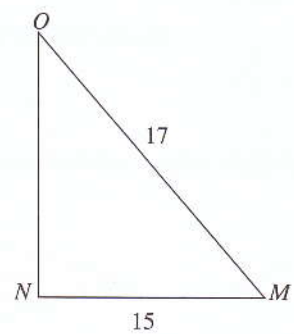
Solution

(a) $NM = \frac{1}{2}\ PQ = 15$ cm

By Pythagoras theorem

$ON^2 = 7^2 - 15^2 = 64$

$\therefore\ ON = \sqrt{64} = 8$ cm



(b) Volume of the pyramid $= \frac{1}{3} \times$ area of base $\times$ height

$= \frac{1}{3} \times 30 \times 30 \times 8$

$= 2400\ \text{cm}^3$

(c) The net of the pyramid is as shown.

Area of $\triangle OQR = \frac{1}{2} \times 30 \times 17 = 255\ \text{cm}^2$

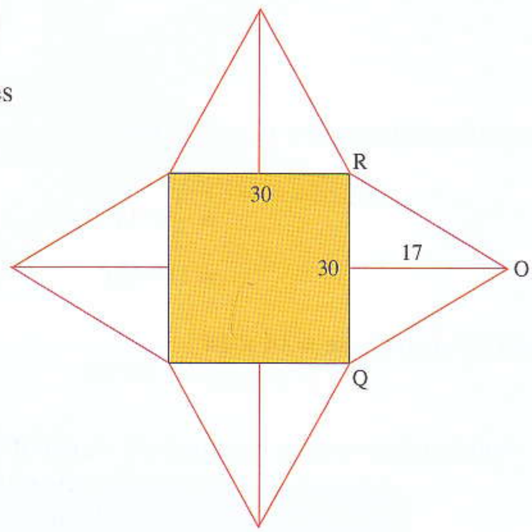
Area of the four faces $= 4 \times 255 = 1020\ \text{cm}^2$

Total surface area of the pyramid

= area of base + area of the four faces

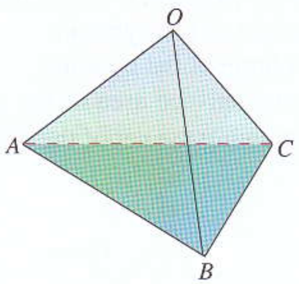
$= 30 \times 30 + 1020$

$= 1920\ \text{cm}^2$

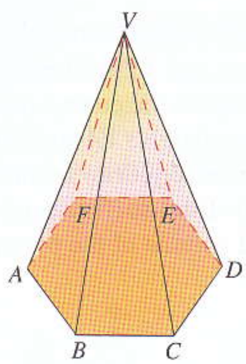


Exercise 7a

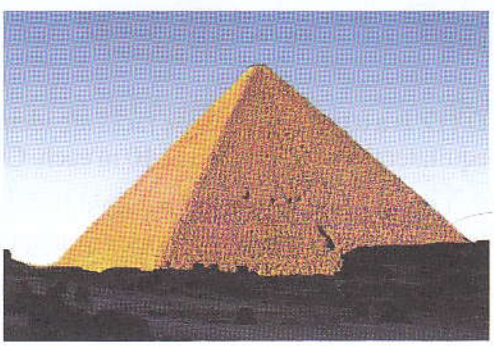
1. *OABC* is a tetrahedron. Given that the area of the base is 15 cm^2 and the height is 4 cm, find the volume of the tetrahedron.



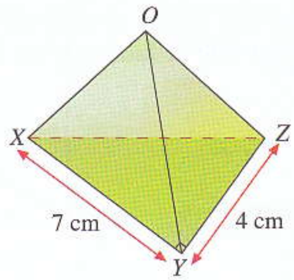
2. *VABCDEF* is a hexagon-based pyramid. Given that the area of the base is 23 cm^2 and the height is 6 cm, find the volume of the pyramid.



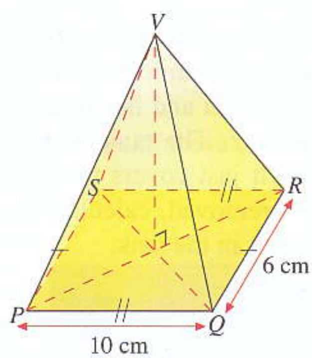
3. The height of the Great Pyramid of Egypt is 146 m and the base is a square of side 229 m. Find the volume of the pyramid, leaving your answer correct to 3 significant figures.



4. *OXYZ* is a pyramid where $X\hat{Y}Z = 90°$, $XY = 7$ cm and $YZ = 4$ cm. Given that the height of the pyramid is 5 cm, find its volume.



5. *VPQRS* is a rectangle-based pyramid where $PQ = 10$ cm and $QR = 6$ cm. Given that the volume of the pyramid is 100 cm^3, find its height.

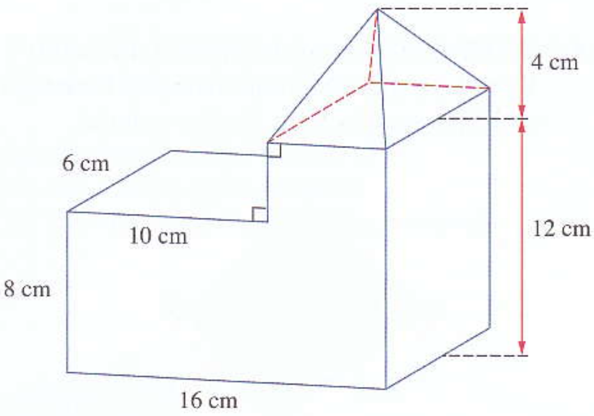


6. Given that the volume of the pyramid with a square base of side 5 m each is 75 m^3, find its height.

7. A pyramid with a triangular base has a volume of 50 cm^3. If the base and height of the triangle are 5 cm and 8 cm respectively, find the height of the pyramid.

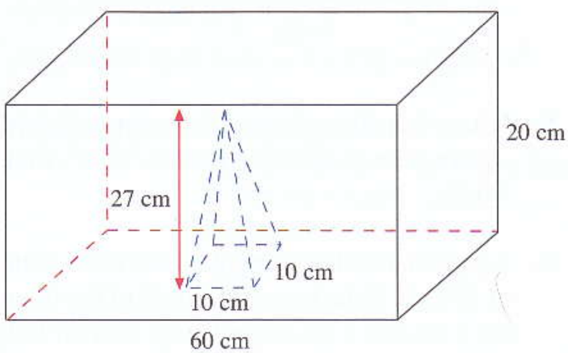
8. The height and volume of a square-based pyramid are 12 cm^3 and 100 cm^3 respectively. Find the length of the square base.

9. Find the volume of the solid shown below.



10. A solid pyramid of height 40 cm with a square base of sides 30 cm each is put into a cubical tank of sides 40 cm each. The tank is then filled with water. If the pyramid is removed, find the depth of water in the tank.

11. A rectangular tank has a base 60 cm by 20 cm. A solid metal pyramid with a square base of sides 10 cm each and height 27 cm is placed inside the tank. The tank is then filled with water until it just covers the pyramid. If the pyramid is removed, calculate the fall in the level of water in the tank.



12. A nickel pyramidal paper weight has a square base of sides 5 cm and a height of 3 cm. What is the mass of a dozen such paper if the mass of 1 cm^3 of nickel is 8.9 g?

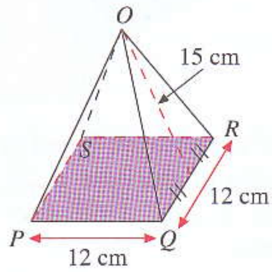
13. A right pyramid has a square base of sides 10 cm each and a slant height of 15 cm. Draw a net of the pyramid and label its dimensions.

14. *OPQRS* is a right pyramid whose base is a square of sides 12 cm each. Given that the slant height of the pyramid is 15 cm. Find

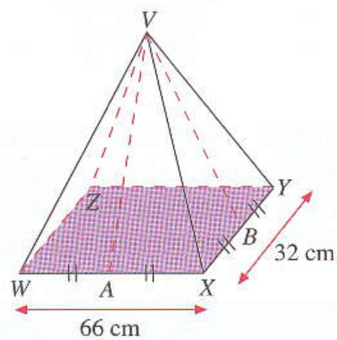
(a) the height of the pyramid,

(b) the volume of the pyramid,

(c) the total surface area of the pyramid by first drawing the net of the pyramid.

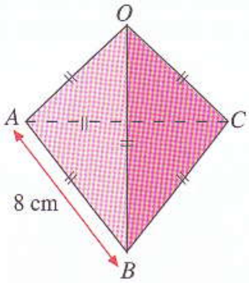


15. *VWXYZ* is a rectangle-based pyramid where $WX = 66$ cm and $XY = 32$ cm. The vertex V is vertically above the centre of the base. Given that the slant heights VA and VB are 56 cm and 63 cm respectively, draw the net of the pyramid and find its total surface area. Also find the height and volume of the pyramid.



16. *OXYZ* is a tetrahedron whose faces are identical equilateral triangles of sides 10 m each. Given that the slant height of the pyramid is $5\sqrt{3}$ m, find its total surface area, leaving your answer correct to 3 significant figures.

17. *OABC* is a tetrahedron whose faces are identical equilateral triangles of sides 8 cm each. Find its slant height, leaving your answer correct to 3 significant figures.



18. *OPQRS* is a right pyramid whose base is a square of side 7 cm. Given that the total surface area of the pyramid is 161 cm^2, find its slant height.

19. *VABCDE* is a right pyramid whose base is a regular pentagon of side 9 m. Given that the area of the base is 139 m^2 and the total surface area of the pyramid is 409 m^2, find its slant height.

Cones

Fig. 7.12 shows some examples of cones. What are some common features of these cones?

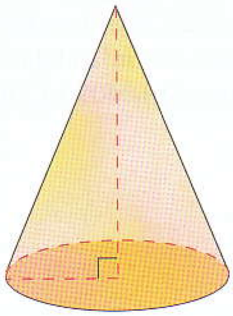
(a) Ice Cream Cone

(b) Party Hat

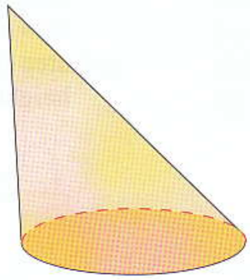
(c) Wheelock Place, Singapore

Fig. 7.12 Examples of Cones

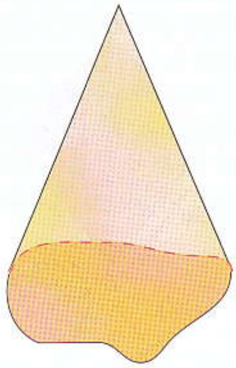
A **cone** is a solid with a vertex and a base that is formed by a **simple closed curve**. If the base is circular, then it is called a circular cone. If the vertex is vertically above the centre of the circular base, it is called a **right circular cone**. Fig. 7.13 shows some examples of cones with different bases.



(a) Circular Base



(b) Elliptical Base



(c) Base is a simple closed curve

Fig. 7.13 Cones with Different Bases

The **perpendicular height** (or height) of a cone is the perpendicular distance from the vertex to the base of the cone (see Fig. 7.14). The **slant height** of a right circular cone is the distance from the vertex to the circumference of the base.

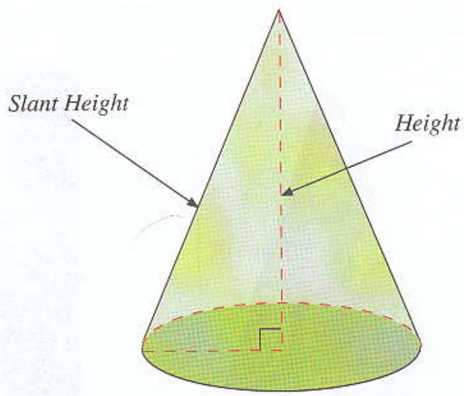
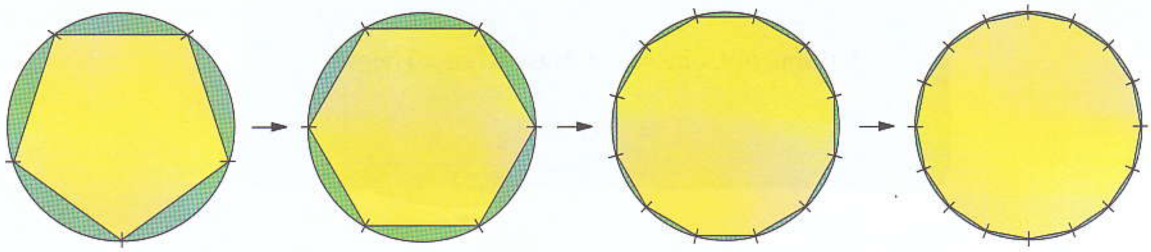


Fig. 7.14 Height and Slant Height

In this chapter, we will study only right circular cones. For simplicity, we will use the term "cone" to mean right circular cone, unless otherwise specified. We will also use the term "height" to mean the perpendicular height of the cone.

Comparison Between Pyramid and Cone

Fig. 7.15 shows (a) a regular pentagon, (b) a regular hexagon, (c) a regular 12-gon and (d) a regular 16-gon. As the number of sides of a regular polygon increases infinitely, what will the polygon become eventually?

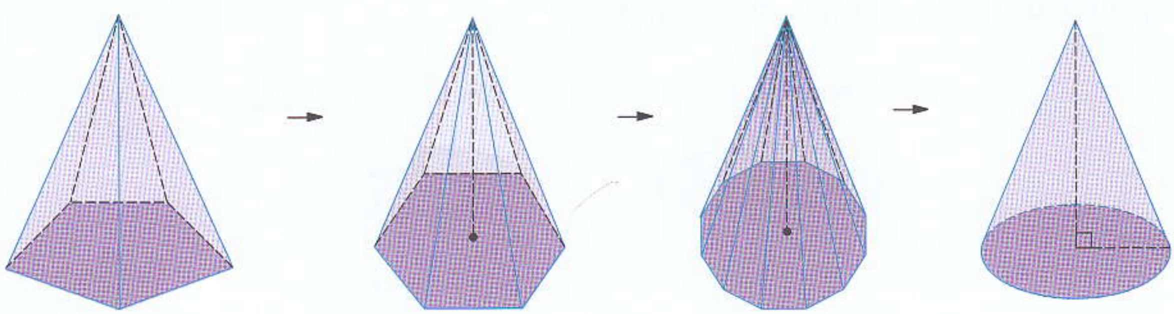


(a) (b) (c) (d)

Fig. 7.15 Sequence of Regular Polygons

Fig. 7.15 shows that the regular polygon will eventually become a circle if the number of the sides of the polygon increases infinitely.

Similarly, as the number of sides of the polygonal base of a right pyramid increases infinitely, the polygonal base will become a circle and hence the pyramid will become a cone (see Fig. 7.16).



(a) (b) (c) (d)

Fig. 7.16 Cones with Different Bases

In many ways, a cone is quite similar to a pyramid. However, **a cone is not a pyramid** because the base of a pyramid must be a polygon which has a **finite** number of sides but the base of a cone is a circle and not a polygon.

Volume of Cones

We have learnt that the volume of a pyramid is equal to $\frac{1}{3} \times \text{base area} \times \text{height}$. We have also observed from Fig. 7.16 that a cone is quite similar to a pyramid. This suggests that the volume of a cone may also be given by:

$$\textbf{Volume of Cone} = \frac{1}{3} \times \textbf{Base Area} \times \textbf{Height}$$
$$= \frac{1}{3}\pi r^2 h$$

Let's do the activity below to confirm the formula for the volume of a cone.

Explorations

We have learnt in Book 1 that the volume of a circular cylinder is $\pi r^2 h$. Consider a cone and a cylinder with the **same circular** base and the **same height** (see Fig. 7.17). If we are to fill the cone with sand and then pour it into the cylinder, what fraction of the cylinder will be filled? If we repeat the process, how many times will it take to fill the cylinder completely?

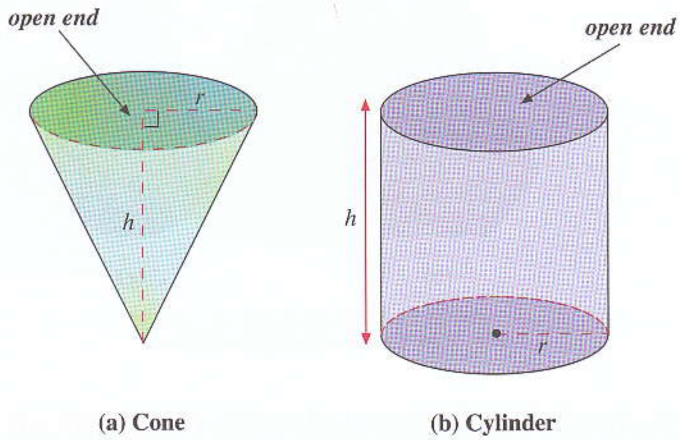
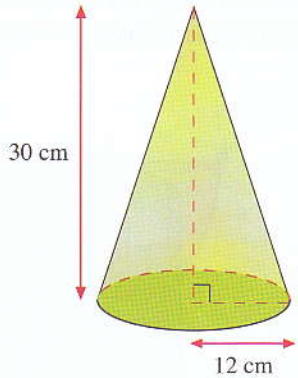


Fig. 7.17 Cone and Cylinder

Example 5

A cone has a circular base of radius 12 cm and a height of 30 cm. Find the volume of the cone, leaving your answer correct to 3 significant figures.



Solution

$$\text{Volume of cone} = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3} \times \pi \times 12^2 \times 30$$
$$= 4520 \text{ cm}^3 \text{ (correct to 3 sig. fig.)}$$

> **Your Attention**
> *For better accuracy, work in terms of π. Convert π to decimal only at the last step.*

Example 6

A cone has a circular base of radius 6 m and a volume of 84π m^3. Find the height of the cone.

Solution

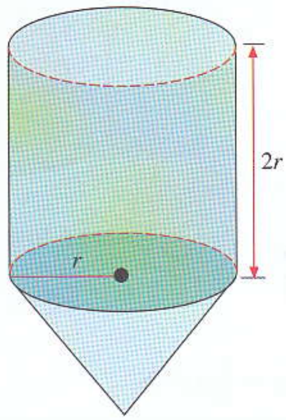
$$\text{Volume of cone} = \frac{1}{3}\pi r^2 h = 84\pi$$
$$\therefore \ = \frac{1}{3} \times \pi \times 6^2 \times h = 84\pi$$
$$h = 7$$

$\therefore$ height of cone = 7 m

Example 7

The following container is made of a hollow cone of internal radius r cm and a right circular cylinder of the same internal radius and height 2r cm. Given that the height of the cone is three-quarters that of the cylinder and 2.7 litres of water is needed to completely fill the conical part of the container, calculate
(a) the amount of water, in litres, needed to completely fill the container,
(b) the total height of the container.

(Take π to be 3.142.)



Solution

Height of cone $= \frac{3}{4} \times$ height of cylinder

$= \frac{3}{4} \times 2r = \frac{3}{2}r$ cm

Volume of cone $= \frac{1}{3}\pi r^2 \times \frac{3}{2}r$

$2700 = \frac{1}{3}\pi r^3$

$\pi r^3 = 5400$

$r = \sqrt[3]{\frac{5400}{3.142}}$

$= 11.98$ cm (correct to 2 decimal places)

(a) Volume of cylinder $= \pi r^2 \times 2r$

$= 2\pi r^3 = 2 \times 5400$

$= 10\ 800\ \text{cm}^3$

$= 10.8$ litres

$\therefore$ the amount of water needed to completely fill the container

$= 2.7 + 10.8 = 13.5$ litres

(b) The total height of the container $= \frac{3}{2}r + 2r$

$= \frac{7}{2}r$

$= \frac{7}{2} \times 11.98$

$= 41.93$ cm

Surface Area of Cone

Fig. 7.18(a) shows a quadrant with radius l. If you fold the quadrant such that the edges VA and VA' are joined together, you will obtain the open cone in Fig. 7.18(b) where the slant height is l and the radius of the circular base is r. How do you find the curved surface area of the cone?

(a) Net of Cone **(b) Cone**

Fig. 7.18 Net of Cone

The curved surface area of the cone is equal to the area of the quadrant (see Fig. 7.18(a). The length of the arc AA' in Fig. 7.18(a) is equal to the circumference of the circular base of the cone in Fig. 7.18(b),

i.e. length of arc $AA' = 2\pi r$.

But length of arc $AA' = \frac{1}{4} \times 2\pi l = \frac{1}{2}\pi l$.

Thus, length of arc $AA' = 2\pi r = \frac{1}{2}\pi l$

$$r = \frac{1}{4} l$$

Therefore, area of quadrant $= \frac{1}{4} \times \pi l^2$

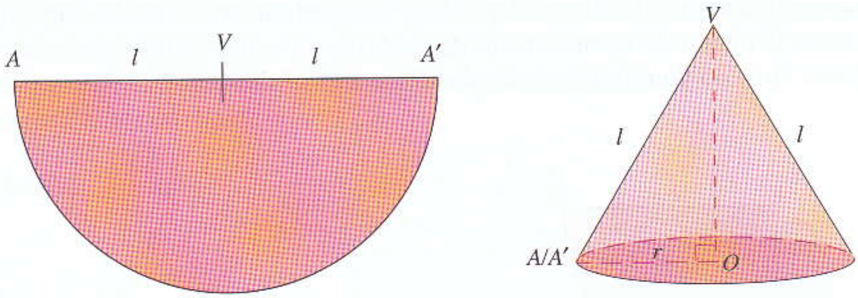
$$= \pi \times \left(\frac{1}{4} l\right) \times l$$

$$= \pi \times r \times l$$

Hence, curved surface area of cone $= \pi \times r \times l$

The above is true if the net of the open cone is a quadrant. What happens if the net is not a quadrant?

Fig. 7.19 (a) shows a semicircle with radius l. If you fold the semicircle such that the edges VA and VA' are joined together, you will obtain the open cone in Fig. 7.19(b) where the slant height is l and the radius of the circular base is r.



(a) Net of Cone **(b) Cone**

Fig. 7.19 Net of Cone

How do you find the curved surface area of the cone?

You can repeat the investigation for open cones with different nets. The above exploration suggests that the curved surface area of a cone is given by

Curved Surface Area of Cone = $\pi r l$

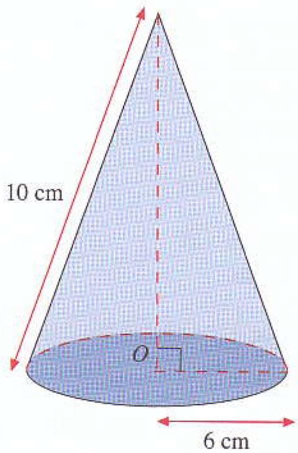
where r is the radius of the circular base and l is the slant height of the cone.

Do it Yourself

Try to give 2 more approaches to get the above formula.

Example 8

A cone has a circular base of radius 6 cm and a slant height of 10 cm. Find the total surface area of the cone, leaving your answer correct to 3 significant figures.



Solution

Total surface area of cone $= \pi r l + \pi r^2$

$= \pi \times 6 \times 10 + \pi \times 6^2$

$= 96\pi$

$= 302 \text{ cm}^2$ (correct to 3 sig. fig.)

Example 9

A cone has a circular base of radius 7 cm and a total surface area of 352 cm^2. Find the slant height of the cone. (Take π to be 3.142.)

Solution

Area of cone $= \pi r l + \pi r^2 = 352$

$3.142 \times 7 \times l + 3.142 \times 7^2 = 352$

$21.994\, l + 153.958 = 352$

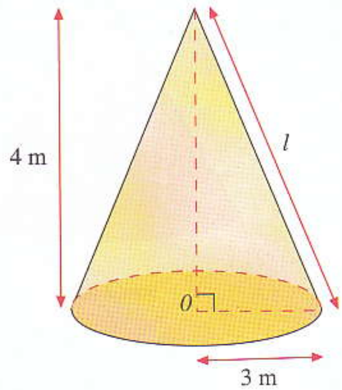
$21.994\, l = 198.042\, l$

$l = 9.00$ (correct to 3 sig. fig.)

$\therefore$ slant height of cone = 9.00 cm

Example 10

A cone has a circular base of radius 3 m and a height of 4 m. Find the curved surface area of the cone.



Solution

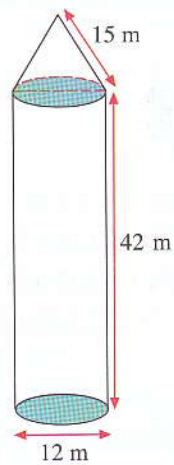
Let the slant height of the cone be l m.

Using Pythagoras' Theorem, $l = \sqrt{3^2 + 4^2}$
$= 5$

$\therefore$ curved surface area of cone $= \pi r l$
$= \pi \times 3 \times 5$
$= 15\pi$
$= 47.1 \text{ m}^2$ (correct to 3 sig. fig.)

Example 11

The figure shows a rocket in the form of a closed cylinder. A cone of the same base radius is attached to the top of the cylinder. Calculate the total surface area of the rocket. (Take π to be 3.142.)



Solution

Area of base of cylinder $= \pi r^2 = \pi \times 6 \times 6 = 36\pi \text{ m}^2$

Area of curved surface of cylinder $= 2\pi r h$
$= 2 \times \pi \times 6 \times 42$
$= 504\pi \text{ m}^2$

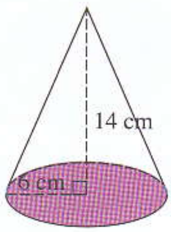
Curved surface area of cone $= \pi r l$
$= \pi \times 6 \times 15 = 90\pi \text{ m}^2$

$\therefore$ total surface area of the rocket $= 36\pi + 504\pi + 90\pi$
$= 630\pi$
$= 630 \times 3.142$
$= 1979.46 \text{ m}^2$

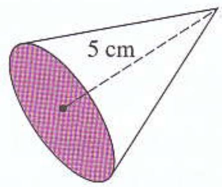
Exercise 7b

1. Find the volume of each of the following cones.

(a)

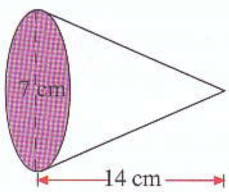


(b)

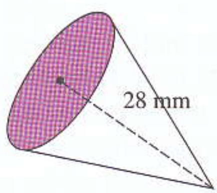


Area of base
= 154 cm ^{2}

(c)



(d)



Circumference of base
= 132 mm

2. A cone has a circular base of radius 8 cm and a volume of 320π cm^3. Find the height of the cone.

3. A cone has a base area of 20 m^2 and a volume of 160 m^3. Find the height of the cone.

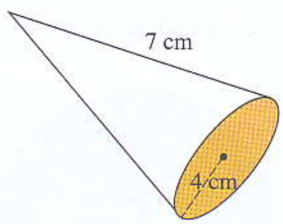
4. A cone has a height of 14 cm and a volume of 132 cm^3. Find the radius of the circular base. (Take π to be 3.142.)

5. A conical funnel of diameter 23.2 cm and depth 42 cm is full of water. If the water is poured into a cylindrical tin of diameter 16.2 cm, find the least possible height of the tin if it must contain all the liquid.

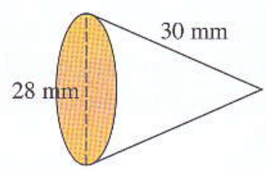
6. A conical block of silver has a height of 16 cm and a base radius of 12 cm. How many coins $\frac{1}{6}$ cm thick and $1\frac{1}{2}$ cm in diameter can be made by melting the silver?

7. Find the curved surface area of each of the following cones.

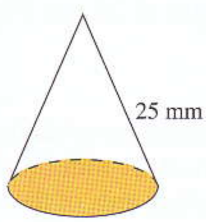
(a)



(b)



(c)



Circumference of base
= 132 mm

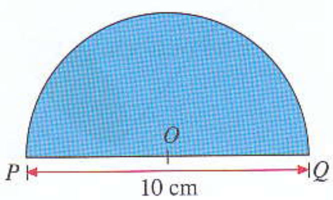
8. A cone has a circular base of radius 14 cm and a total surface area of 1012 cm^2. Find the slant height of the cone. (Take π to be 3.142.)

9. A cone has a circular base of radius 6 mm and a curved surface area of 84π mm^2. Find the slant height of the cone.

10. An open cone has a slant height of 5 cm and a curved surface area of 251 cm^2. Find the radius of its circular base, leaving your answer correct to 3 significant figures.

11. An open cone has a circular base of radius 10 cm and a slant height of 20 cm. Draw a net of the cone and label its dimensions.

12. The semicircle shown is folded to form a right circular cone so that the arc PQ becomes the circumference of the base. Find
    (a) the diameter of the base,
    (b) the curved surface area of the cone.

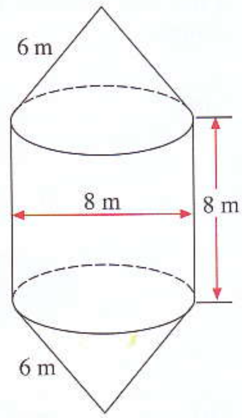


13. A cone has a circular base of radius 5 cm and a height of 12 cm. Find the curved surface area of the cone.

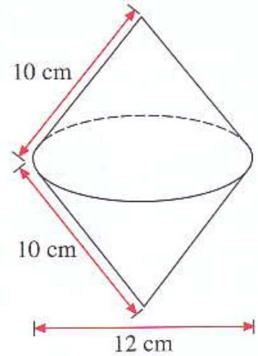
14. A cone has a circular base of radius 8 cm and a slant height of 20 cm. Find the volume of the cone.

15. A circular cone has a height of 17 mm and a slant height of 21 mm. Find the volume and the total surface area of the cone.

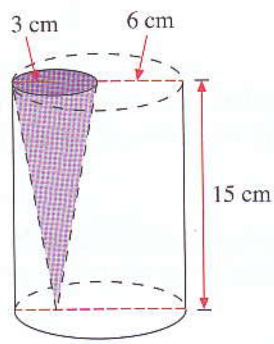
16. Calculate
    (a) the total surface area,
    (b) the volume of the solid cylinder with conical ends as shown below.



17. Find
    (a) the volume,
    (b) the total surface area of the solid with conical ends as shown in the figure on the right.



18. The figure below shows a cylinder of diameter 12 cm and height 15 cm. Ahole in the shape of a cone is bored into one of its ends. If the cone has a diameter equal to half of the diameter of the cylinder, find the volume of the remaining solid.



Volume of Sphere

Fig. 7.20 Examples of Spheres

Fig. 7.20 shows some examples of spheres. What are their volumes?

There is an interesting story about how to find the volume of an object.

Archimedes is one of the three greatest mathematicians of all times. He lived during 287-212 BC in Greece. One day, the King asked a goldsmith to make him a gold crown. After the crown was made, the King doubted whether the crown was really made of pure gold. So he tasked Archimedes to find out. It was easy to find the mass of the crown by weighing it. The problem was to find its volume. Archimedes thought for a few days but he still had no idea. Then, he went to take a bath. As he stepped into the bath full of water, the water overflowed. This gave Archimedes an idea of how to find the volume of the crown. He was so excited that he dashed out into the street shouting "Eureka!", meaning "I have found it!" But he forgot that he had not put on his clothes.

What Archimedes found out was that a sinking solid displaces an amount of water equal to the volume of the solid. So, to find the volume of the crown, all you need to do is to fill up the container shown in Fig. 7.21(a). It is called a Eureka can, named after the famous incident described in the story above. Then you put in the crown and the volume of water displaced will be equal to the volume of the crown, as shown in Fig. 7.21(b).

Suppose Archimedes found out that the volume of water displaced was 714 cm^3 and the mass of the crown was 11.6 kg. Given that the density of gold is 19.3 g/cm^3 (i.e. the mass of 1 cm^3 of gold is 19.3g) determine whether the crown was made of pure gold.

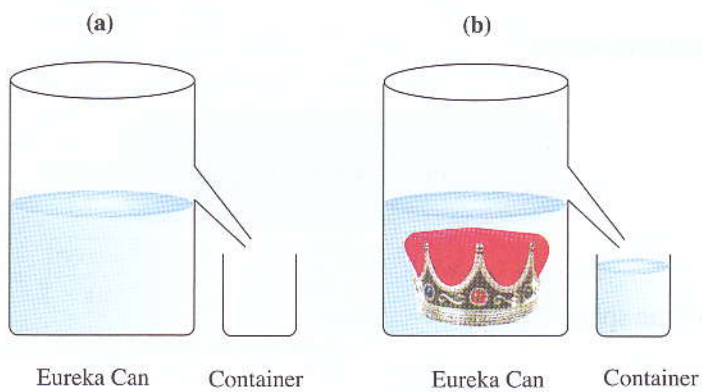
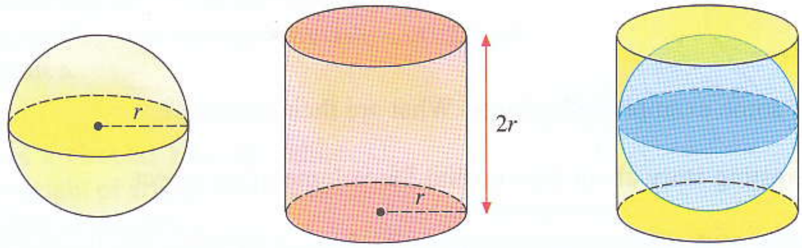


Fig. 7.21 Finding Volume of Crown

Using the same technique, Archimedes discovered a formula to calculate the volume of a sphere. Fig. 7.22 shows (a) a sphere of radius r and (b) a circular cylinder of base radius r and height $2r$. Archimedes filled the cylinder with water and put the sphere into it (see Fig. 7.22(c)).



(a) (b) (c)

Fig. 7.22 Finding Volume of Sphere

Archimedes found that the volume of water displaced was equal to $\frac{2}{3}$ of the volume of the cylinder. This was one of his greatest discoveries.

Volume of Cylinder = $\pi r^2 h$ = ______________________________

Volume of Sphere = $\frac{2}{3}$ × Volume of Cylinder = ________________________________

From the above Exploration, we get

$$\text{Volume of Sphere} = \frac{4}{3}\pi r^3$$

where r is the radius of the sphere.

Example 12

A ball bearing (which is spherical in shape) has a radius of 0.3 cm.

(a) Calculate the volume of the ball bearing.

(b) Find the mass of 6000 identical ball bearings if they are made of steel of density 7.85 g/cm³.

Solution

(a) Volume of ball bearing $= \frac{4}{3}\pi r^3$

$= \frac{4}{3}\pi (0.3)^3$

$= 0.113\ \text{cm}^3$ (correct to 3 sig. fig.)

(b) Mass of 6000 ball bearings = Volume of 6000 ball bearings × Density

$= 6000 \times 0.1131 \times 7.85$

$= 5330$ g (correct to 3 sig. fig.)

Pay Attention

To be accurate to 3 sig. fig. in the final answer, intermediate working must be corrected to 4 sig. fig.

Example 13

A basketball has a volume of 5600 cm³. Find its radius.

Solution

Volume of basketball $= \frac{4}{3}\pi r^3$

$$r^3 = \frac{5600 \times 3}{4\pi}$$

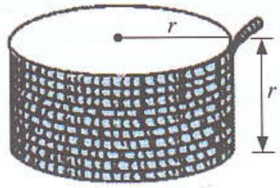
$$r = \sqrt[3]{\frac{5600 \times 3}{4\pi}}$$

$= 11.0$ cm (correct to 3 sig. fig.)

Surface Area of Sphere

Archimedes also discovered a formula for the surface area of a sphere. Fig. 7.23 (a) shows half a sphere, called a hemisphere, with radius r. One end of a piece of twine is stuck in the centre of the curved surface of the hemisphere by a pin and then coiled around the curved surface completely. Fig. 7.23 (b) shows a circular cylinder with base radius r and height r. One end of a piece of twine is stuck at the bottom of the curved surface of the cylinder by a pin and then coiled around the curved surface completely.

(a) Hemisphere



(b) Cylinder

Fig. 7.23 Finding Surface Area of Sphere

Archimedes found that the two pieces of twine were of the same length.

Length of second piece of twine = $2\pi rh$ = ________________________

Curved surface of sphere = $2 \times$ length of first piece of twine
= $2 \times$ length of second piece of twine
= $2 \times$ ______________
= ________________

Therefore,

Surface Area of Sphere = $4\pi r^2$

where r is the radius of the sphere.

Example 14

A solid sphere has a diameter of 14 cm. Calculate its surface area. (Take π to be 3.142.)

Solution

Radius of sphere $= \frac{14}{2} = 7$ cm

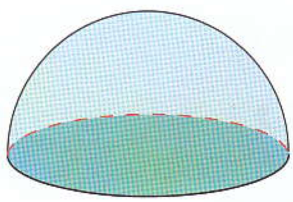
Surface area of sphere $= 4\pi r^2$

$= 4 \times 3.142 \times 7^2$

$= 616\ \text{cm}^2$ (correct to 3 sig. fig.)

Example 15

A hemisphere has a curved surface area of 175 cm^2. Find its radius.



Solution

Curved surface area of hemisphere $= \frac{1}{2} \times 4\pi r^2 = 2\pi r^2 = 175$

$r = \frac{175}{2\pi}$

$r^2 = \sqrt{\frac{175}{2\pi}}$

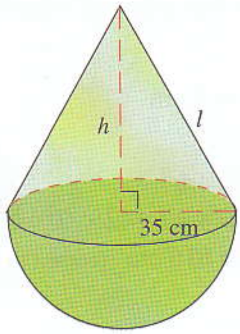
$= 5.28$ cm (correct to 3 sig. fig.)

Example 16

The figure shows a solid made up of a right circular cone fastened on top of a hemisphere of equal radius

of 35 cm. Given that the volume of the cone is equal to $1\frac{1}{5}$ the volume of the hemisphere. Find

(a) the height of the cone,
(b) the total surface area of the solid in terms of π.



Solution

(a) Let the height of the cone be h cm.

Volume of cone = $\frac{1}{3}\pi(35)^2h$ cm^3.

Volume of hemisphere = $\frac{2}{3}\pi(35)^3$ cm^3.

$$\therefore \quad \frac{1}{3}\pi(35)^2h = \frac{6}{5}\left(\frac{2}{3}\pi(35)^3\right)$$

$$h = \frac{\frac{12}{15}\pi(35)^3}{\frac{1}{3}\pi(35)^2} = 84 \text{ cm}$$

(b) Let the slant height of the cone be l cm.

$$\therefore \quad l = \sqrt{35^2 + 84^2} = 91 \text{ cm.}$$

Curved surface area of cone = $\pi(35)(91) = 3185\pi$ cm^2

Curved surface area of hemisphere = $2\pi(35)^2 = 2450\pi$ cm^2

$\therefore$ Total surface area of the solid = $3185\pi + 2450\pi$

$= 5635\pi$ cm^2.

Example 17

A solid metal ball of radius 3 cm is melted and the metal obtained is recast to form a solid circular cone of radius 4 cm. Find the height of the cone.

Solution

$$\text{Volume of ball} = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi (3)^3 = 36\pi \text{ cm}^3$$

$$\text{Volume of cone} = \frac{1}{3}\pi R^2 h = 36\pi$$

$$\therefore \frac{1}{3}\pi (4)^2 h = 36\pi$$

$$\frac{16}{3}\pi h = 36\pi$$

$$h = \frac{36\times 3}{16} = 6.75 \text{ cm}$$

The height of the cone is 6.75 cm.

Exercise 7c

1. Find the volume of a sphere with the following radius:
   (a) 8 cm
   (b) 14 mm
   (c) 4 m

2. Calculate the mass of 5000 spherical lead shots each of diameter 0.4 cm if the mass of 1 cm^3 of lead is 11.3 g.

3. Find the number of steel ball bearings, each of diameter 0.7 cm, that can be made from 1 kg of steel, given that 1 cm^3 of steel weighs 7.85 g.

4. A hollow metal sphere has an internal radius of 20 cm and an external radius of 30 cm. Given that the density of the metal is 7.8 g/cm^3, find the mass of the sphere, expressing your answer in kg .

5. Find the radius of a sphere with the following volume:
   (a) $1416\ \text{cm}^3$
   (b) $12\,345\ \text{mm}^3$
   (c) $780\ \text{m}^3$

6. Find the radius of a sphere with the following volume:
   (a) $972\pi\ \text{cm}^3$
   (b) $498\pi\ \text{mm}^3$
   (c) $15\frac{3}{16}\pi\ \text{m}^3$

7. Fifty-four solid hemispheres, each of diameter 2 cm, are melted to form a single sphere. Find the radius of the sphere.

8. Find the surface area of a sphere with the following radius:
   (a) 12 cm
   (b) 9 mm
   (c) 3 m

9. Find the total surface area of a hemisphere of radius 7 cm. (Take π to be 3.142.)

10. Calculate the radius and surface area of a sphere of volume $850\ \text{m}^3$.

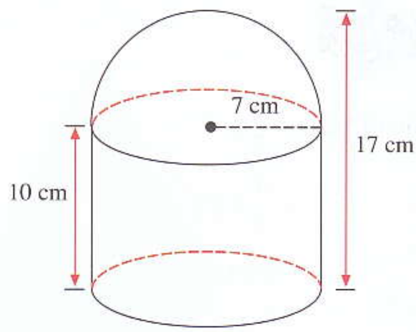
11. Find the radius of a sphere with the following surface area:
   (a) $210\ \text{cm}^2$
   (b) $7230\ \text{mm}^2$
   (c) $3163\ \text{m}^2$

12. Find the radius of a sphere with the following surface area:
   (a) $64\pi\ \text{cm}^2$
   (b) $911\pi\ \text{mm}^2$
   (c) $49\pi\ \text{m}^2$

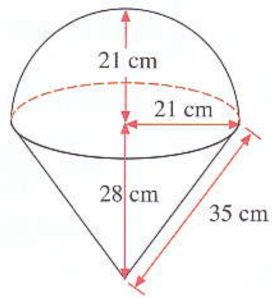
13. A basketball has a surface area of $1810\ \text{cm}^2$. Find the radius of the basketball and the volume of air in it.

14. Find the volume and surface area of the following solids. (Take π to be 3.142.)

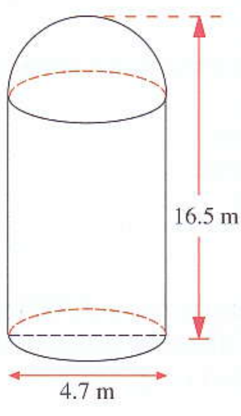
(a)



(b)



15. A storage tank consists of a hemisphere of diameter 4.7 m and a cylinder. The overall height of the tank is 16.5 m. Find the capacity of the tank.

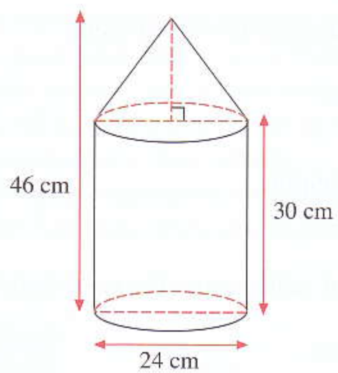


16. The diagram below shows a solid consisting of a cone and a cylinder with a common base. Find the

(a) volume of the solid,

(b) total surface area of the solid.

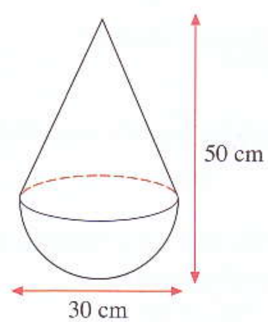
(Leave your answer in terms of π.)



17. The following diagram shows a solid consisting of a right circular cone fastened to a hemisphere with a common base. Find the

(a) volume of the solid,

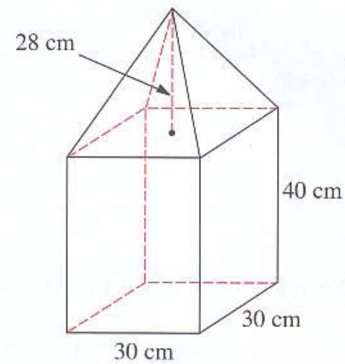
(b) total surface area of the solid.



18. The figure shows a solid consisting of a pyramid of height 28 cm fastened to a cuboid of height 40 cm and a square base of sides 30 cm each. Find

(a) the volume of the solid,

(b) the total surface area of the solid. (Leave your answer correct to 2 decimal places).



19. A solid metal ball of radius 2 cm is melted and the metal obtained recast to form a solid right circular cone of radius 5 cm. Find the height of the cone.

20. A sphere of diameter 25 cm is half-full with acid, all of which is drained into a tall cylindrical beaker 16 cm in diameter. What is the depth of the acid in the beaker?

21. A cylindrical tin has an internal diameter of 18 cm. It contains water to a height of 13.2 cm. When a heavy spherical ball of diameter 9.3 cm is immersed in it, what is the new height of the water level?

22. A cylindrical can has a horizontal base of radius 3.4 cm. It contains sufficient water so that when a sphere is placed inside, the water just covers the sphere. If the sphere fits exactly into the can, calculate

(a) the total surface area of the can in contact with the water when the sphere is inside,

(b) the depth of the water in the can before the sphere was put inside.

SUMMARY

1. For a pyramid and a prism with the same polygonal base and the same height,

$$\text{Volume of Pyramid} = \frac{1}{3} \times \text{Volume of Prism}$$

$$= \frac{1}{3} \times \text{Base Area} \times \text{Height}$$

2. For a cone and a cylinder with the same circular base of radius r and the same height h,

$$\text{Volume of Cone} = \frac{1}{3} \times \text{Volume of Cylinder}$$

$$= \frac{1}{3} \times \text{Base Area} \times \text{Height}$$

$$= \frac{1}{3}\pi r^2 h$$

3. Total Surface Area of Pyramid = Sum of Areas of All the Faces

4. For a cone with circular base of radius r and slant height l,

   Curved Surface Area of Cone $= \pi rl$

   Total Surface Area of Cone $= \pi rl + \pi r^2 = \pi r(l + r)$

5. For a sphere with radius r,

   Volume of Sphere $= \dfrac{4}{3}\pi r^3$

   Surface Area of Sphere $= 4\pi r^2$

Review Examples 7

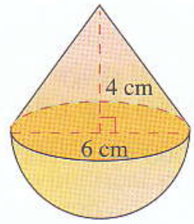
Example 1

The figure shows a solid consisting of a cone and a hemisphere with a common base. The cone has a height of 4 cm and a base diameter of 6 cm.

(a) Find the volume and surface area of the solid.

(b) The solid is melted and recast to form a cylinder with a height of 4 cm. Find the radius of the cylinder.

(c) If 1000 of these cylinders are to be painted and each tin of paint is enough to paint 5 m^2, how many tins of paint are needed to paint these cylinders?



Solution

(a) Volume of solid = Volume of cone + Volume of hemisphere

$$= \frac{1}{3}\pi r^2 h + \frac{2}{3}\pi r^3$$

$$= \frac{1}{3}\pi (3)^2(4) + \frac{2}{3}\pi(3)^3$$

$$= 30\pi$$

$$= 94.2 \text{ cm}^2 \quad \text{(correct to 3 sig. fig.)}$$

Slant height of cone, $l = \sqrt{4^2+3^2}$ (using Pythagoras' Theorem)

$$= 5 \text{ cm}$$

Surface area of solid = Curved surface area of cone + Surface area of hemisphere

$$= \pi r l + 2\pi r^2$$

$$= \pi(3)(5) + 2\pi(3)^2$$

$$= 33\pi$$

$$= 104 \text{ cm}^2 \quad \text{(correct to 3 sig. fig.)}$$

(b) Volume of cylinder = $\pi r^2 = 30\pi$

$$\therefore \pi r^2 (4) = 30\pi$$

$$\therefore r^2 = \frac{30}{4} = \frac{15}{2}$$

$$\therefore r = \sqrt{\frac{15}{2}}$$

$$= 2.74 \text{ cm} \quad \text{(correct to 3 sig. fig.)}$$

(c) Surface area of one cylinder $= 2\pi rh + 2\pi r^2$

$= 2\pi\sqrt{7.5}(4) + 2\pi(7.5)$

$= 116.0 \text{ cm}^2$ (correct to 4 sig. fig.)

Surface area of 1000 cylinders $= 116\,000 \text{ cm}^2$

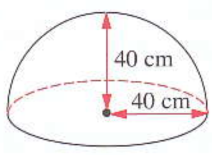
$= 11.60 \text{ m}^2$ (correct to 4 sig. fig.)

Since $\frac{11.60}{5} = 2.32$, then 3 tins of paint are needed to paint the 1000 cylinders.

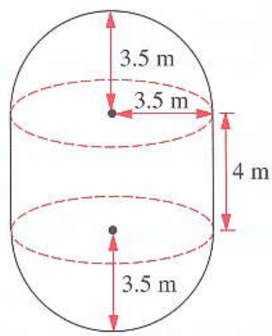
You cannot round 2.32 tins to the nearest whole number because 2 tins of paint are not enough.

Review Questions 7

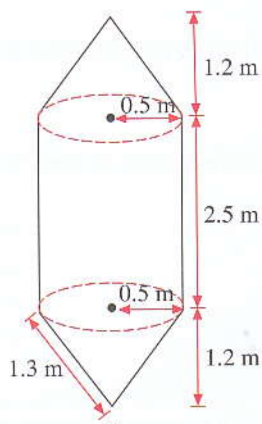
1. Find the volume and the total surface area of the following solids.
(Take π to be 3.142.)



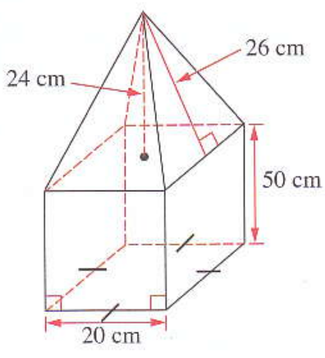
(a)



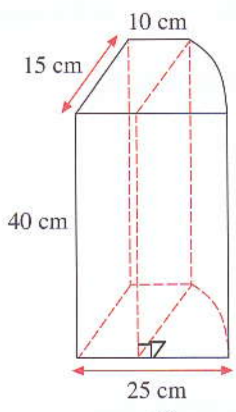
(b)



(c)

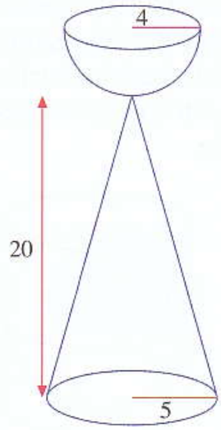


(d)



(f)

2.



The diagram shows a decorative structure made up of a cone and a hemisphere. The hemisphere has a radius of 4 cm. The base radius of the cone is 5 cm and the height of the cone is 20 cm.

(a) Calculate the curved surface area of the outside of the cone.

(b) Calculate the volume of the solid cone.

(c) Calculate the curved surface area of the outside of the hemisphere.

(d) Some liquid is poured into the hemisphere to a depth of x centimetres. Show that the volume of the liquid is $\frac{x^3}{64}$ of the volume of the hemisphere.

(Take π to be 3.142.)

3. (Take π to be 3.142 and correct the final results to 3 decimal places)

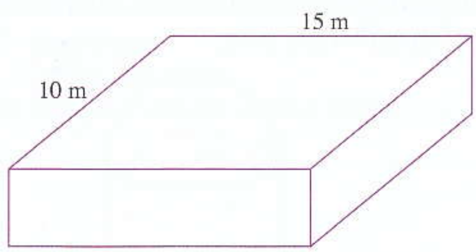


Diagram I

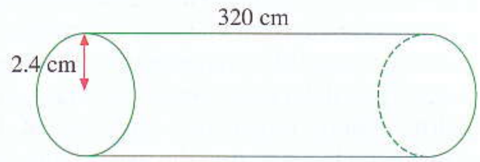


Diagram II

(a) Diagram I shows a rectangular swimming pool of length 15 m and width 10 m. The pool was completely filled with 600 m^3 of water. Calculate the depth of the pool.

(b) Diagram II shows a cylindrical pipe of radius 2.4 cm and length 320 cm. Calculate

(i) the total area, in square metres, of the outside of the open cylindrical pipe,

(ii) the volume, in litres, of water that can flow through the pipe at any moment of time.

(c) The pipe is used to drain the pool of water completely. Calculate how long would it take for the pool to be completely emptied of water if it takes 4 seconds for water to flow through the whole length of the pipe. Give your answer to the nearest hour.

4. Find the cost of painting a hemispherical roof 10 m in diameter at \$1.50 per square metre, leaving your answer correct to the nearest cent.

5. A mound of earth is shaped like a right circular cone 6 m high with a base circumference of 30 m. Find the cost of removing it at 99 cents per cubic metre. (Take π to be 3.142)

6. Forty solid hemispheres, each of diameter 2 cm, are melted to form a solid cone with base diameter 6 cm. Find the height of the cone.

7. Two solid spheres have surface areas 144π cm^2 and 256π cm^2 respectively. They are melted and recast to form a larger sphere. Find the surface area of this sphere in cm^2.

8. The external diameter of a hollow metal sphere is 12 cm and its thickness is 2 cm. Find the radius of a solid sphere made of the same material and having the same mass as the hollow sphere. Given that the mass of 1 cm^3 of the metal is 5.4 g, find the mass of the sphere in kg.

9. In an experiment, a small spherical drop of oil is allowed to fall onto the surface of water so that it produces a thin film of oil covering a large area.

(a) Given that the volume of a drop of oil is 12.5 mm^3, find the number of drops which can be produced by 5000 mm^3 of oil.

(b) Given that the volume V of a sphere of radius r is $\frac{4}{3}\pi r^3$, express r in terms of V and π. Then calculate the radius of one drop of oil.

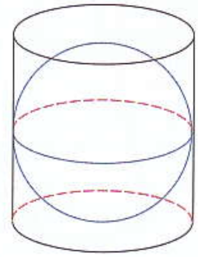
10. A cylinder and a cone have the same height, $2r$, and base diameter, $2r$. A sphere has a diameter $2r$. Find the ratio of the volume of the cylinder to that of the cone and to that of the sphere.

11. Washing dishes under a running tap wastes an average of 155 litres of water per wash. Calculate the total amount of water wasted in a year if a family washes dishes twice a day for a year with 365 days. If this water were to be transferred into small spherical containers of radius 8.5 cm, how many such containers will be needed?

12. Using full-flushes instead of half-flushes on cisterns in the toilet wastes 9.5 litres of water per flush. A school has a population of 1380. Assuming that each person uses the toilet three times in a day and a full-flush is used each time, calculate the total amount of water wasted in the course of the year if a year has only 205 school days, giving your answer in m^3.

If the wasted water were to be put into cylindrical containers with a base radius of 24 cm and height of 38 cm, calculate the number of such containers that need to be used.

13. The figure shows a sphere of radius r fitting exactly into a cylinder, i.e., the sphere touches the cylinder at the top, bottom and curved surface. Show that the surface area of the sphere is equal to the area of the curved surface of the cylinder.



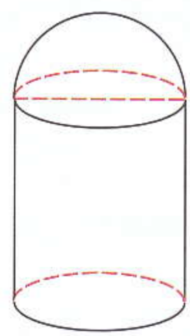
14. A solid metal model of a rocket is made by fastening a cone of vertical height 49 cm and base radius 18 cm to a circular cylinder of length 192 cm and radius 18 cm. If the mass of the model is 2145 kg, calculate the density of the metal in kg/m^3, giving your answer correct to the nearest whole number.

15. A bowl is made by cutting into half a hollow sphere of external diameter 50.8 cm, made of metal 2.54 cm thick.

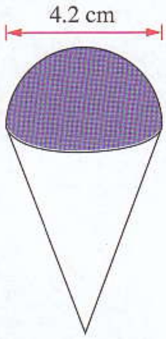
(a) If the bowl is completely filled with a liquid of density 31.75 kg/m^3, calculate the mass of the liquid in grams.

(b) The bowl when empty weighs 97.9 kg. Calculate the density, in kg/m^3, of the metal of which the bowl is made of.

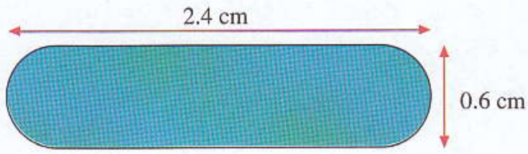
16. The diagram shows a solid cylindrical stone pillar whose top is a hemisphere. Given that the pillar is 40 cm in diameter and has the same mass as a solid stone sphere of the same material, with radius 40 cm, find the height of the pillar.

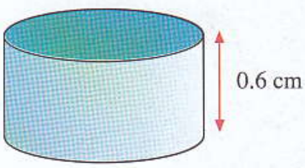


17. The wafer cone shown contains 56 cm^3 of ice-cream filled to the bottom. The diameter of the cone is 4.2 cm, and the top of the ice-cream has the shape of a hemisphere. Find the height of the cone .



18. A vitamin tablet is 2.4 cm long and is in the shape of a cylinder with hemispheres of diameter 0.6 cm attached to both ends. Another vitamin tablet is in the shape of a cylinder of height 0.6 cm.

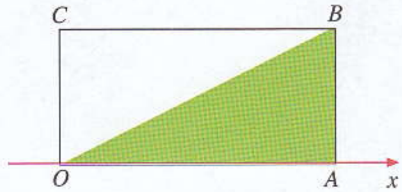




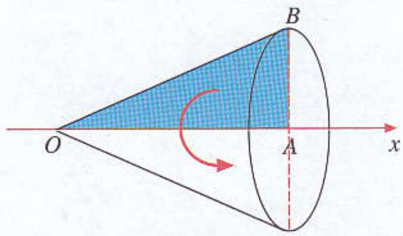
(a) Find the radius of the cylindrical tablet given that its surface area is equal to that of the first tablet.

(b) Find the volume of each tablet.

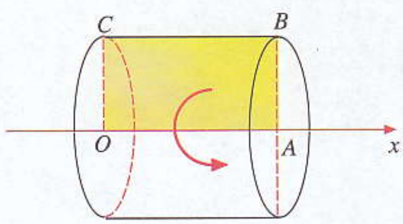
***19.** Figure **(a)** below shows a triangle *OAB* and rectangle *OABC*. Figure **(b)** shows a cone formed by rotating the triangle *OAB* about the x-axis. Figure **(c)** shows a cylinder formed by rotating the rectangle *OABC* about the x-axis. Notice that the cone and the cylinder have the same circular base and the same height. Since the area of the rectangle *OABC* is 2 times the area of the triangle *OAB*, why is it that the volume of the cylinder is not 2 times the volume of the cone?



(a) *Triangle & Rectangle*



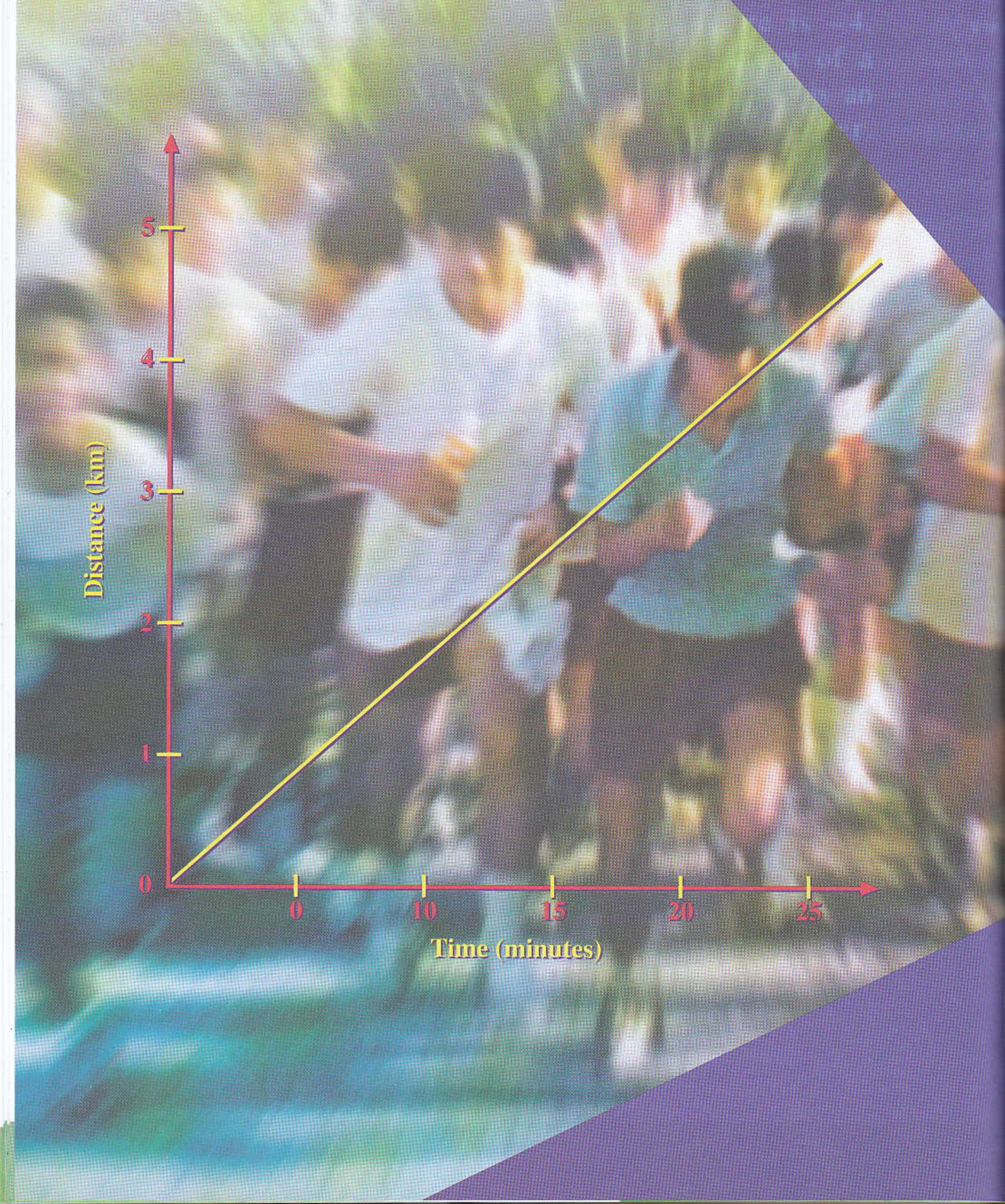
(b) *Cone*



(c) *Cylinder*

In this chapter, you will learn how to

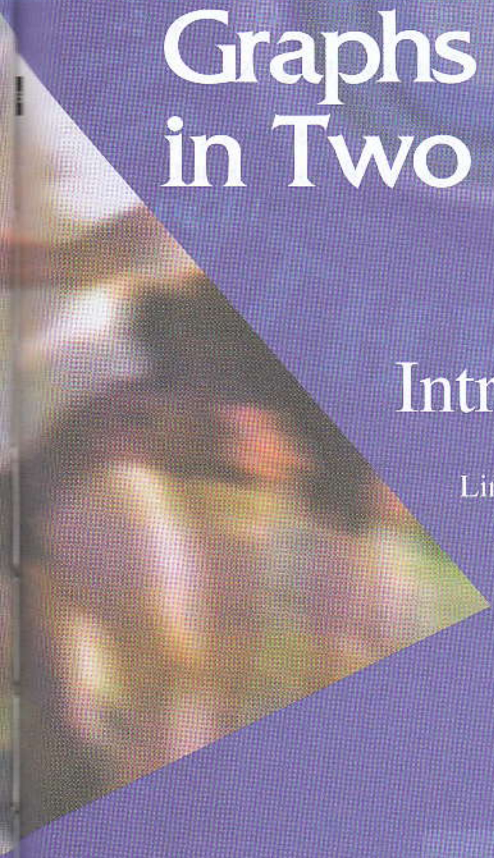
- *plot straight line graphs;*
- *solve simultaneous linear equations graphically.*



CHAPTER

8

Graphs of Linear Equations in Two Unknowns



Introduction

Linear graphs are used in a lot of daily situations. For example, we use a travel graph to show a journey undertaken by a moving objects, like a moving vehicle. The picture displays a linear graph of a runner.

In Book 1, we learnt how to plot coordinate points (or ordered pairs) on a piece of graph paper and how to draw the graph of a linear equation in two variables, e.g. $y = 2x$ and $y = 2x + 1$. Now we shall continue learning how to draw graphs of linear equations.

Choice of Appropriate Scales for Graphs

Before we proceed to draw a graph, we have to choose a suitable scale. The following guidelines are useful:

1. Use a convenient scale for both the x- and y-axes. For example, 1 cm to represent 1 unit, 2 units, 4 units, 5 units or 10 units. Avoid using awkward scales like 1 cm to represent 2.3 units or 1 cm to represent 7 units.

2. The scale used for the x-axis need not be the same as that used for the y-axis. You can use, for example, 1 cm to represent 2 units on the x-axis and 1 cm to represent 5 units on the y-axis.

3. Choose a suitably large scale so that the graph will be more than half the size of the given graph paper. The bigger the graph, the more accurate the results obtained from it will be. Look at the largest and smallest values of x and do a rough calculation to decide the scale that would give the largest possible graph. Do the same for the y values.

Explorations

1. Using a suitable scale, draw the graph of $y = 3x - 1$. Compare your graph with your classmates'. Do they look different? Why?

2. From your graph, find
   (a) the values of y when $x = -0.8$ and 1.3,
   (b) the values of x when $y = -2.8$, 0 and 0.8.

Using the scale 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, we draw the graph of $y = 3x - 1$ as shown below.

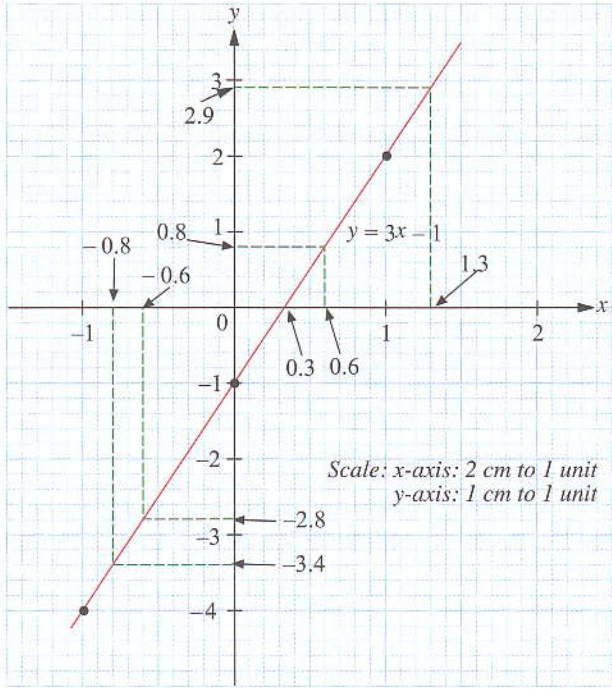


Fig. 8.1

To find the value of y when $x = -0.8$, we draw a vertical line from the horizontal axis where $x = -0.8$ to meet the graph and then continue from the graph horizontally to meet the y-axis. Read off the value of y from the y-axis. From the graph, $y = -3.4$ when $x = -0.8$. Similarly, $y = 2.9$ when $x = 1.3$.

To find the value of x when $y = -2.8$, we draw a horizontal line from the vertical axis where $y = -2.8$ to meet the graph and continue from the graph vertically to meet the x-axis. Read off the value of x from the x-axis. From the graph, $x = -0.6$ when $y = -2.8$. Similarly, $x \approx 0.3$ when $y = 0$ and $x = 0.6$ when $y = 0.8$.

Thinking Time

How can we check the accuracy of the above results? We can do so by substitution and solving the equation $y = 3x - 1$. If your results are inaccurate, how do you improve your graph?

Exercise 8a

1. For each question, complete the table. Plot the coordinates and draw the graph of the equation.

(a) $y = x$

x	0	1	2	3	4
$y = x$					

(b) $y = x + 3$

x	−2	−1	0	1	2
$y = x + 3$					

(c) $y = -2x$

x	−1	0	1	2	3
$y = -2x$					

(d) $y = x - 2$

x	−3	−2	−1	0	1
$y = x - 2$					

2. Using the values of x from −1 to 3, construct a table showing some points whose coordinates satisfy each of the following equations. Plot the points and draw the graph of the equations.

(a) $y = x + 2$
(b) $y = -x$
(c) $y = x - 1$
(d) $y = 2x + 1$

3. **(a)** Given the equation $y = 3x + 5$, copy and complete the table below.

x	−1	0	1
$y = 3x + 5$			

(b) Plot the points and draw a straight line through the points.
(c) From the graph, find
  (i) the values of y when $x = -2$, 0.6 and 1.5,
  (ii) the values of x when $y = -1$, 0.8 and 2.9.

4. **(a)** Given the equation $y = 4x$, copy and complete the table below.

x	−1	0	1
$y = 4x$			

(b) Draw the graph of the equation $y = 4x$.
(c) From the graph, find
  (i) the values of y when $x = -0.5$, 1.5 and 2.5,
  (ii) the values of x when $y = -2$, 1.6 and 3.6.

Graphs of Equations of the Form $y = c$

On a piece of graph paper, using a scale of 2 cm to represent 1 unit on both axes for values of x and y from -6 to 6, draw the graphs of the following equations:

Equation	Three points that lie on the line
$y = 2$	$(0, 2)$ $(4, 2)$ $(-3, 2)$
$y = 5$	(,) (,) (,)
$y = -1$	(,) (,) (,)
$y = -4$	(,) (,) (,)

(i) What do you notice about the vertical changes of all the lines?
(ii) What is the gradient of the line of the form $y = c$?
(iii) What does the line $y = 0$ represent?
(iv) Write down your observations about the line of the form $y = c$.

From the above Exploration, we notice that $y = 2$, $y = 5$, $y = -1$ and $y = -4$ are horizontal straight lines parallel to the x-axis (See Fig. 8.2).

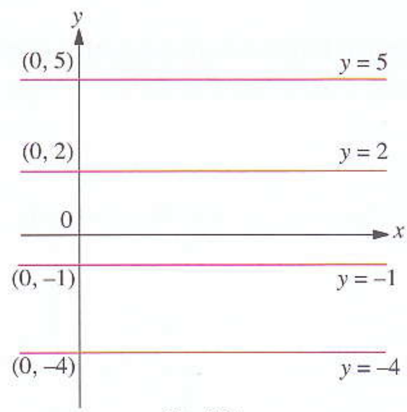
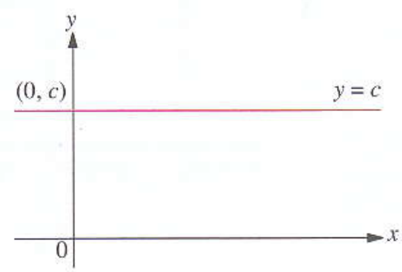


Fig. 8.2

In general, the graph of the equation $y = c$ is a line passing through the point $(0, c)$ and parallel to the x-axis. The gradient of the line of the form $y = c$ is zero.



Graphs of Equations of the Form $x = a$

Explorations

On a piece of graph paper, using a scale of 2 cm to represent 1 unit on both axes for values of x and y from -6 to 6. Draw the graphs of the following equations:

Equation	Three points that lie on the line
$y = 2$	$(2, -2)$ $(2, 0)$ $(2, 2)$
$y = 5$	(,) (,) (,)
$y = -1$	(,) (,) (,)
$y = -4$	(,) (,) (,)

(i) What do you notice about the horizontal changes of all the lines?
(ii) What is the gradient of the line of the form $x = a$?
(iii) What does the line $x = 0$ represent?
(iv) Write down your observations about the line of the form $x = a$.

From the above Exploration, we notice that $x = 2$, $x = 5$, $x = -1$ and $x = -4$ are vertical straight lines parallel to the y-axis (See Fig. 8.3).

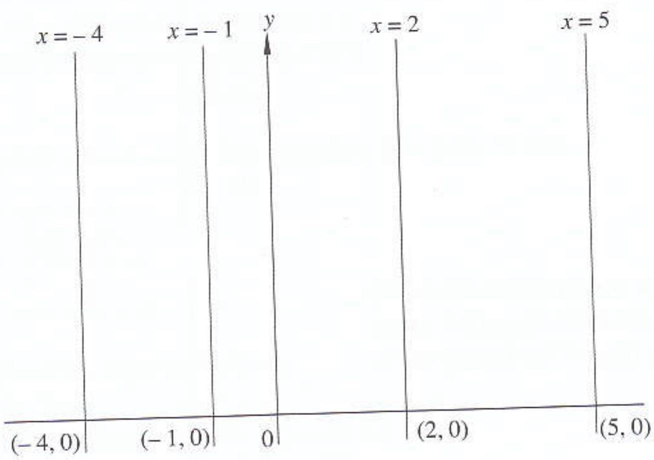
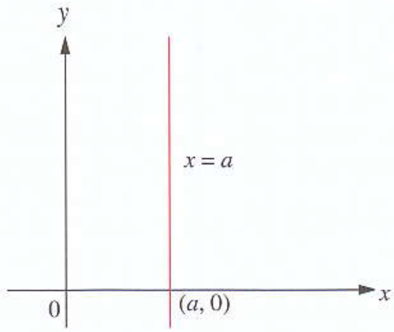


Fig. 8.3

In general, the graph of the equation $x = a$ is a line passing through the point $(a, 0)$ and parallel to the y-axis. The gradient of the line of the form $x = a$ is undefined.



Exercise 8b

1. Each of the following sets of points lies on a line. Write down the equation of the line.

(a) $(2, 2), (-1, 2), (-3, 2), (5, 2), (10, 2), (0, 2)$.

(b) $(-6, 9), (-2, 9), (4, 9), (-1, 9), (7, 9), (9, 9)$.

(c) $(0, -3), (-5, -3), (1, -3), (8, -3), (-8, -3), \left(-1\frac{1}{2}, -3\right)$.

(d) $\left(-\frac{1}{3}, -\frac{1}{2}\right), \left(4, -\frac{1}{2}\right), \left(-4, -\frac{1}{2}\right), \left(-7, -\frac{1}{2}\right), \left(8, -\frac{1}{2}\right)$.

(e) $(10, 0), \left(4\frac{1}{2}, 0\right), (-6, 0), (-9, 0), (0, 0)$.

2. Write down the equations of the lines on which the following points lie.

(a) $(12, -5), (12, 6), (12, -9), (12, -3), (12, 10), (12, 5)$.

(b) $(5, 4), \left(5, \frac{1}{2}\right), (5, -6), (5, -11), (5, 0), (5, 12)$.

(c) $(-4, 4), (-4, 0), (-4, -10), (-4, 8), (-4, -4), (-4, 7)$.

(d) $(0, -8), (0, 6), \left(0, \frac{1}{2}\right), (0, 0), (0, 16), (0, -15)$.

(e) $\left(-\frac{1}{4}, 9\right), \left(-\frac{1}{4}, \frac{1}{2}\right), \left(-\frac{1}{4}, -4\right), \left(-\frac{1}{4}, -10\right), \left(-\frac{1}{4}, 0\right), \left(-\frac{1}{4}, -\frac{1}{4}\right)$.

3. State the equations of the lines on which the following points lie.

(a) $(6, -6), (-8, -6), (15, -6), (1, -6), (-9, -6)$.

(b) $(3, 9), (3, 27), (3, -8), (3, -81), (3, 0)$.

(c) $(-10, 2), (-10, -1), (-10, -10), (-10, 10), (-10, 5)$.

(d) $(4, 8), (8, 8), (-7, 8), (-14, 8), (9, 8)$.

(e) $\left(-2, -\frac{1}{3}\right), \left(4, -\frac{1}{3}\right), \left(\frac{2}{3}, -\frac{1}{3}\right), \left(-4\frac{1}{2}, -\frac{1}{3}\right), \left(10, -\frac{1}{3}\right)$.

4. Write down the equation of each horizontal line in Fig. 8.4, including the x-axis.

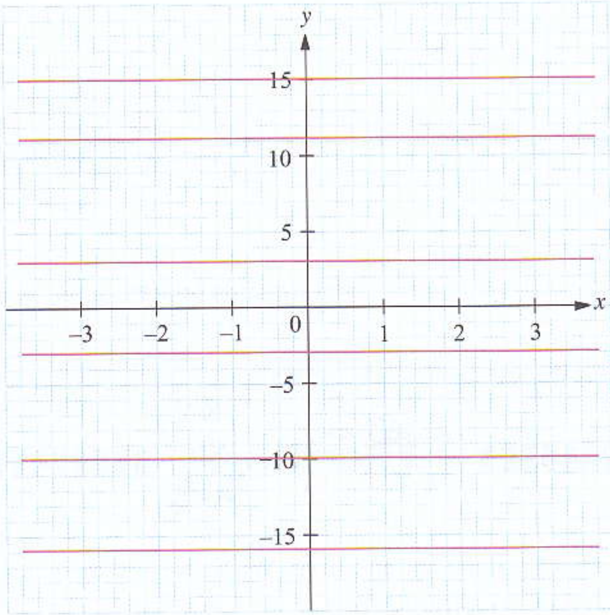


Fig. 8.4

5. Write down the equation of each vertical line in Fig. 8.5, including the y-axis.

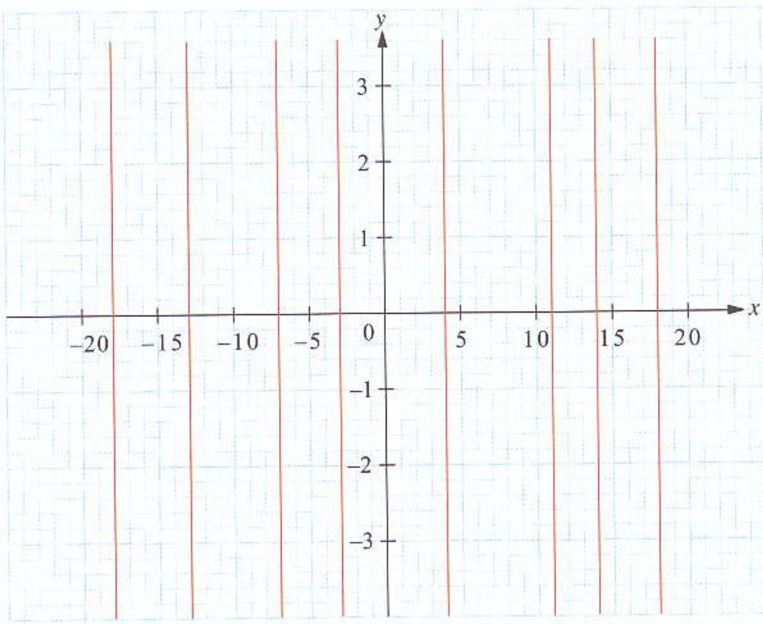


Fig. 8.5

6. Write down the coordinates of four points on each of the following lines:

(a) $y = 3\frac{1}{2}$
(b) $x = -6$
(c) $x = \frac{2}{5}$
(d) $y = -7$
(e) $y = 4.2$
(f) $x = -3.3$
(g) $x = 20$
(h) $y = -16$

7. Draw the graphs of the following equations:

(a) $x = 3$
(b) $y = 10$
(c) $y = -4$
(d) $y = -7$
(e) $x = 5$
(f) $x = -5$
(g) $y = 3.5$
(h) $x = -2.5$
(i) $x = 7.5$
(j) $x = 5\frac{1}{2}$
(k) $y = 8$
(l) $y = -\frac{4}{3}$

Graphs of Equations of the Form $y = mx$

Explorations

Work in pairs for this activity.
You may use graph paper or computer software (e.g. Graphmatica or Winplot).

1. On the same piece of graph paper and using the same axes and scales, draw the graphs of

(a) $y = x$,
(b) $y = 2x$,
(c) $y = 3x$,
(d) $y = -2x$,
(e) $y = -4x$.

Is there any common feature among the five lines?

2. (a) Draw the graph of each of the following equations on the same piece of graph paper:

(i) $y = x$
(ii) $y = 2x$
(iii) $y = -x$
(iv) $y = -2x$
(v) $y = \frac{1}{2}x$
(vi) $y = -\frac{1}{4}x$
(vii) $y = -3x$
(viii) $y = \frac{3}{4}x$
(ix) $y = 6x$

(b) Does each graph pass through the origin?
(c) Find the gradient of each line.

From the above activity, we notice that the graph of $y = mx$, where m is a constant, is a straight line passing through the origin and with the gradient of m.

Fig. 8.6 shows a series of graphs of the form $y = mx$ where m is positive. Do you notice that each line **rises** from **left** to **right**? What is the relationship between the steepness of a line and the numerical value of m?

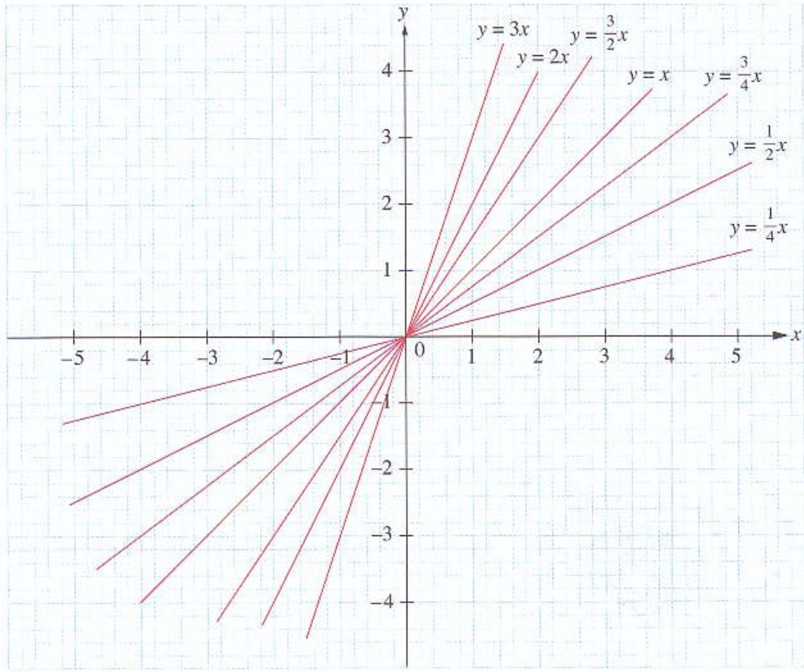


Fig 8.6

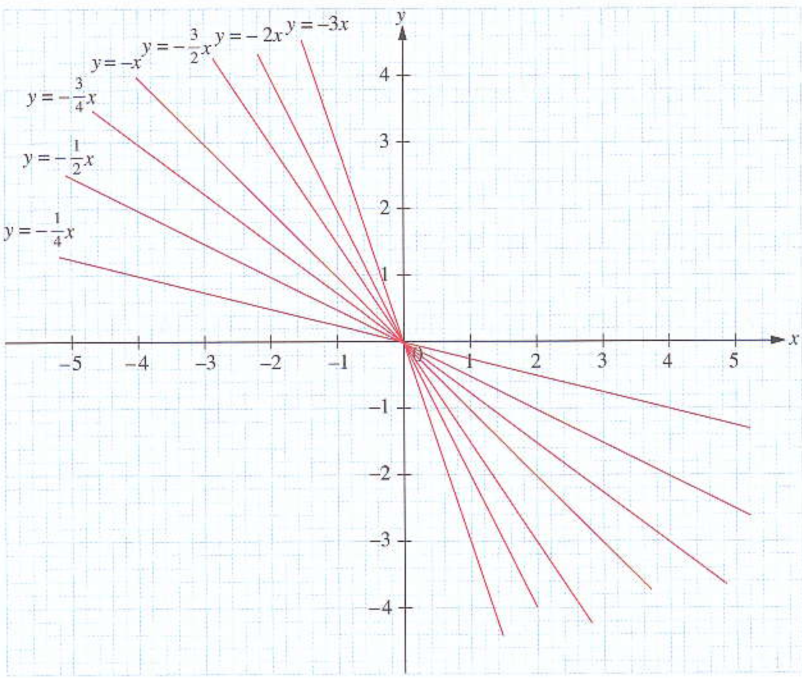


Fig 8.7

Fig. 8.7 shows another series of graphs of the form $y = mx$ where m is negative. Do you notice that the lines **fall** from **left** to **right**? What is the relationship between the steepness of a line and the numerical value of m?

Graphs of Equations of the Form $y = mx + c$

Explorations

You may work in pairs for this activity using graph paper or computer software (e.g Graphmatica or Winplot).

1. (a) Draw the graphs of the following equations on the same graph paper:

(i) $y = 5 - 2x$ **(ii)** $y = 4x + 3$ **(iii)** $y = -5 - 3x$ **(iv)** $y = -\frac{1}{2}x + 1$

(v) $y = 3x - 2$ **(vi)** $y = -\frac{3}{2}x + \frac{5}{2}$

(b) Does each graph pass through the origin?

(c) At what point does each graph cut the y-axis?

2. On the same piece of graph paper and using the same scales and axes, draw the graphs of

(a) $y = 2x$, **(b)** $y = 2x + 2$, **(c)** $y = 2x + 4$, **(d)** $y = 2x - 3$,
(e) $y = 2x - 5$.

What do you observe from these lines? Are there any similarities?

3. On the same piece of graph paper and using the same scales and axes, draw the graphs of

(a) $y = -4x$, **(b)** $y = -4x + 5$, **(c)** $y = -4x + 1$, **(d)** $y = -4x - 2$,
(e) $y = -4x - 5$.

What similarities do the lines have?

4. On the same piece of graph paper and using the same axes and scales, draw the graphs of

(a) $y = x + 3$, **(b)** $y = 2x + 3$, **(c)** $y = 3x + 3$, **(d)** $y = -x + 3$,
(e) $y = -2x + 3$, **(f)** $y = -3x + 3$.

Are there similarities among the six lines?

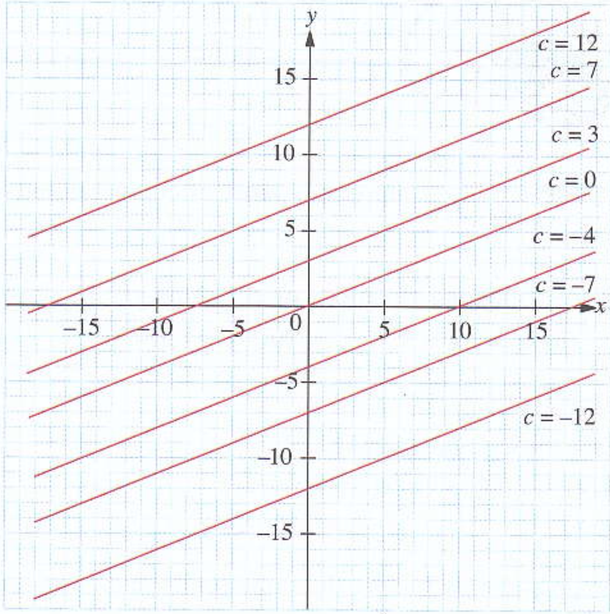
5. On the same piece of graph paper and using the same scales and axes, draw the graphs of

(a) $y = x - 2$, **(b)** $y = 2x - 2$, **(c)** $y = 3x - 2$, **(d)** $y = -x - 2$,
(e) $y = -2x - 2$.

What similarities do the lines have?

From questions 2 and 3 of the above Exploration, we notice that if m remains the same while c takes on different values, the graphs of equations of the form $y = mx + c$ are parallel lines cutting the y-axis at points with coordinates given by $(0, c)$.

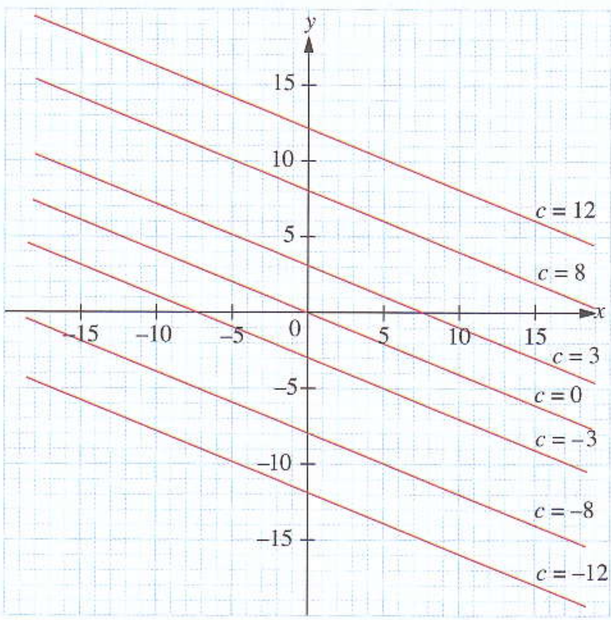
Fig. 8.8 shows a series of graphs of the form $y = mx + c$ where m has a constant positive value of $\frac{2}{5}$ while c takes on different values. The lines are parallel and rise from left to right.



$y = mx + c$ where $m = \frac{2}{5}$ (> 0)

Fig. 8.8

Fig. 8.9 shows another series of graphs of the form $y = mx + c$. Here m has a constant negative value of $-\frac{2}{5}$ while c takes on different values. The lines are parallel but fall from left to right.

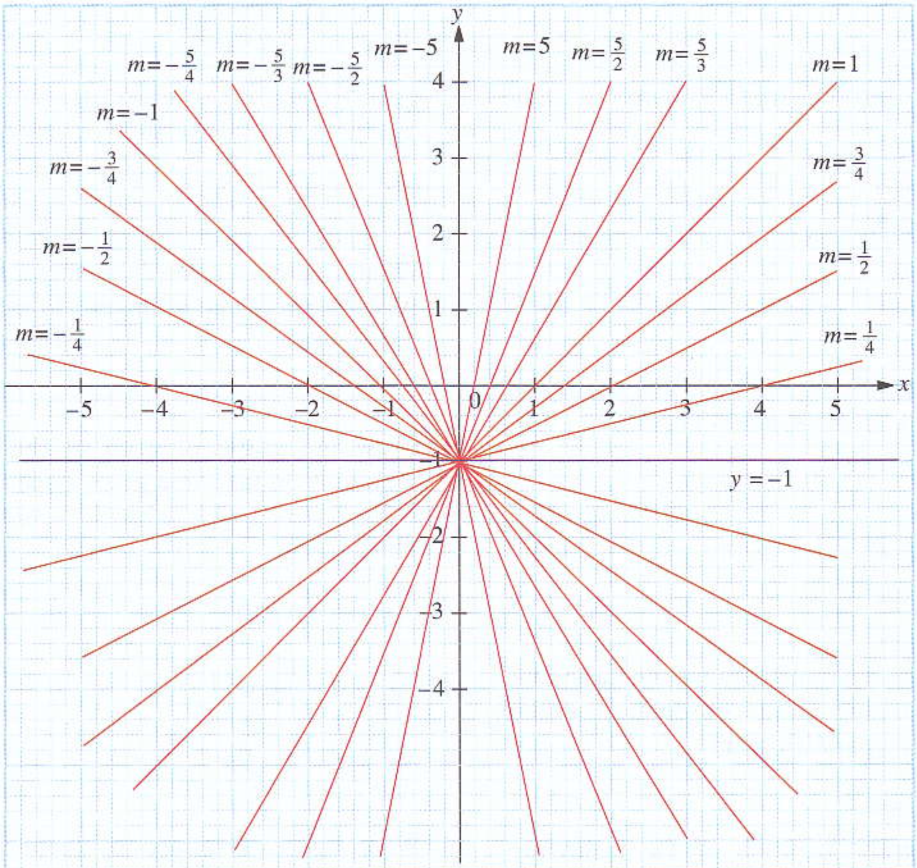


$y = mx + c$ where $m = -\frac{2}{5}$ (< 0)

Fig. 8.9

From Questions 4 and 5 of the Exploration, we notice that graphs of equations of the form $y = mx + c$ all pass through the point $(0, c)$ if c remains unchanged while m varies.

Fig. 8.10 shows a series of graphs of $y = mx + c$ where c takes a constant value of -1 while m takes on different values. The diagram shows an interesting pattern of graphs passing through a common point $(0, -1)$.



$y = mx + c$ where $c = -1$

Fig. 8.10

Exercise 8c

1. On separate diagrams, draw the graphs of the following equations and state the gradient of each line:
   (a) $2y = 3x$
   (b) $3y = -5x + 2$
   (c) $3y = 5x - 3$
   (d) $2y + 3x = 7$
   (e) $4y + x = 2$
   (f) $x + 2y = 0$
   (g) $y = 4x - 8$
   (h) $y = -\frac{1}{3}x$

2. Draw the graph of each of the following equations on the same graph paper:
   (a) $y = 2$
   (b) $y = 6$
   (c) $y = 2x - 2$
   (d) $y = 2x - 6$

   Do you obtain a parallelogram from these four lines? Write down the coordinates of the vertices of the parallelogram.

3. Draw the graph of each of the following equations on the same graph paper:
   (a) $y = -x + 6$
   (b) $y = x - 2$
   (c) $y = -x + 10$
   (d) $y = x + 2$

   What figure is formed by these four lines? Write down the coordinates of the vertices of this figure.

4. Draw the graph of each of the following equations on the same graph paper:
   (a) $y = \frac{1}{3}x$
   (b) $y = \frac{1}{3}x + 4$
   (c) $y = -\frac{1}{3}x + 4$
   (d) $y = -\frac{1}{3}x + 8$

   What figure is formed by these four lines? Write down the coordinates of the vertices of this figure.

5. Draw the graph of each of the following equations on the same graph paper:
   (a) $y = -\frac{3}{2}x + 9$
   (b) $y = \frac{1}{2}x + 5$
   (c) $y = -\frac{1}{2}x + 11$
   (d) $y = \frac{3}{2}x - 9$

   What figure is formed by these four lines? Write down the coordinates of the vertices of this figure.

Solving Simultaneous Linear Equations Using Graphical Method

Explorations

1. Using the elimination or substitution method that you learnt in Chapter 5, solve the simultaneous equations $2x + 3y = 5$ and $3x - y = 2$.

2. Draw the graphs of the linear equations $2x + 3y = 5$ and $3x - y = 2$ on the same rectangular plane. Then using the graphs, find the coordinates of their intersection point.

Is the pair of values of x and y you have obtained for (2) the same as the solution you have obtained in (1)?

Fig 8.11 shows the graphs of $2x + 3y = 5$ and $3x - y = 2$. The graphs intersect at the point (1, 1), ie. $x = 1$, $y = 1$. It is the same as the solution of the above simultaneous equations.

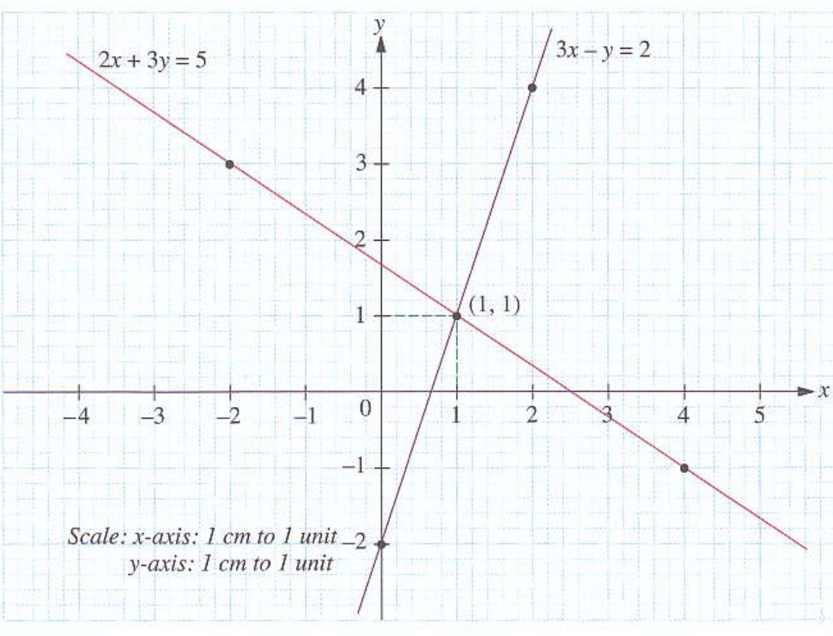
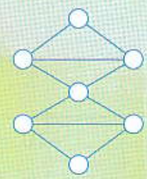


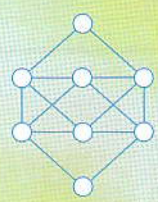
Fig. 8.11

Just for fun

1. Fill in the 7 circles with the digits 1, 2, 3, 4, 5, 6 and 7 so that no two consecutive integers are joined by a segment.



2. Fill in the circles below with the digits 1, 2, 3, 4, 5, 6, 7 and 8 so that no two consecutive integers are joined by a segment.



Example 1

Solve the simultaneous equations

$2x - 5y = 32,$
$2x + 3y = 0$ *graphically.*

Solution

Construct a table of values for each equation.

$2x - 5y = 32$

x	-4	1	6
y	-8	-6	-4

$2x + 3y = 0$

x	-3	0	3
y	2	0	-2

Fig. 8.12 shows that the graphs intersect at the point $(6, -4)$. Thus the solution of the simultaneous linear equations is $x = 6$ and $y = -4$.

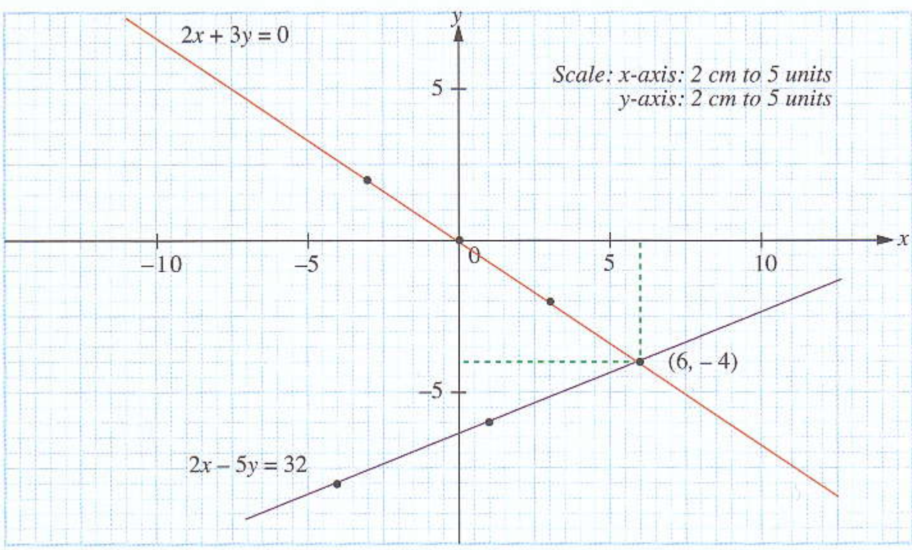


Fig 8.12

Explorations

You may work in pairs for this activity using graph paper or computer software (e.g. Graphmatica or Winplot).

Notes

Multiplying $x + y = 1$ throughout by 3, we obtain $3x + 3y = 3$. These two equations are identical.

1. **(a)** Draw the graphs of the following pairs of equations:

(i) $x + y = 1,\ 3x + 3y = 3$;

(ii) $2x + 3y = -1,\ 20x + 30y = -10$;

(iii) $x - 2y = 5,\ 5x - 10y = 25$.

(b) What do you notice about the graphs of the pairs of simultaneous equations?

(c) What are the solutions if you solve the simultaneous equations graphically?

2. **(a)** Draw the graphs of the following simultaneous equations:

(i) $x + y = 1,\ 4x + 4y = 20$;

(ii) $2x + 3y = -1,\ 20x + 30y = -40$;

(iii) $x - 2y = 5,\ 5x - 10y = 30$.

(b) What do you notice about the graphs of the pairs of simultaneous equations?

(c) What are the solutions if you solve the simultaneous equations graphically?

From the above Exploration, we conclude that :

In part (1), the graphs of each of the three sets of equations are identical, i.e. the two lines coincide. Thus, the graphs have an infinite number of common points. Therefore, the simultaneous equations have an infinite number of solutions.

In part (2), the graphs of each of the three sets of equations are parallel lines. Thus they do not intersect and have no common points. Therefore, the simultaneous equations have no solution.

Exercise 8d

Solve the following simultaneous equations using graphical method:

1. $3x - y = 0$
$2x - y = 1$

2. $3x + y = 2$
$2x - y = 3$

3. $x - y = -3$
$x - 2y = -1$

4. $4x + y = 2$
$4x + y = -3$

5. $x + 2y = 3$
$2x + 4y = 6$

6. $3x - 2y = 7$
$2x + 3y = 9$

7. $x + 4y = 12$
$4x + y = 18$

8. $3x + 2y = 4$
$5x + y = 2$

9. $3x - 4y = 10$
$5x + 7y = 3$

10. $2x + 5y = 25$
$3x - 2y = 9$

11. $3x - 2y = 13$
$2x + 2y = 0$

12. $3x - 4y = 25$
$4x - 5y = 32$

SUMMARY

1. A graph is a drawing which shows the relationship between numbers or quantities.

2. Graphs of linear equations are straight lines.

3.

Equation	Graph
$y = c$	Parallel to the x-axis and the gradient is 0
$x = a$	Parallel to the y-axis
$y = mx$	Passes through the origin and has the gradient, m
$y = mx + c$	Cuts the y-axis at the point $(0, c)$ and has the gradient, m

4. The solution of simultaneous linear equations lies at the point of intersection of their graphs.

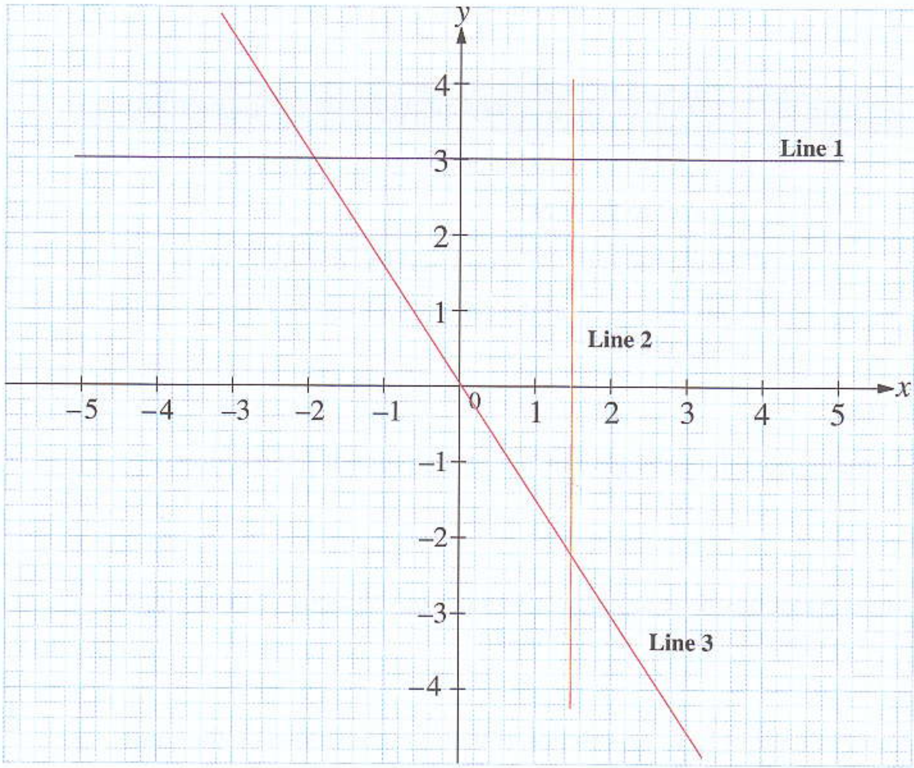
5. Simultaneous linear equations have an infinite number of solutions if their graphs drawn on the same rectangular plane are identical.

6. Simultaneous linear equations have no solution if their graphs drawn on the same rectangular plane are parallel.

Review Examples 8

Example 1

Write down the equations of the lines 1, 2 and 3 in the graph.



Solution

The equation of Line 1 is $y = 3$.

The equation of Line 2 is $x = 1.5$.

The equation of Line 3 is $y = -\frac{3}{2}x$.

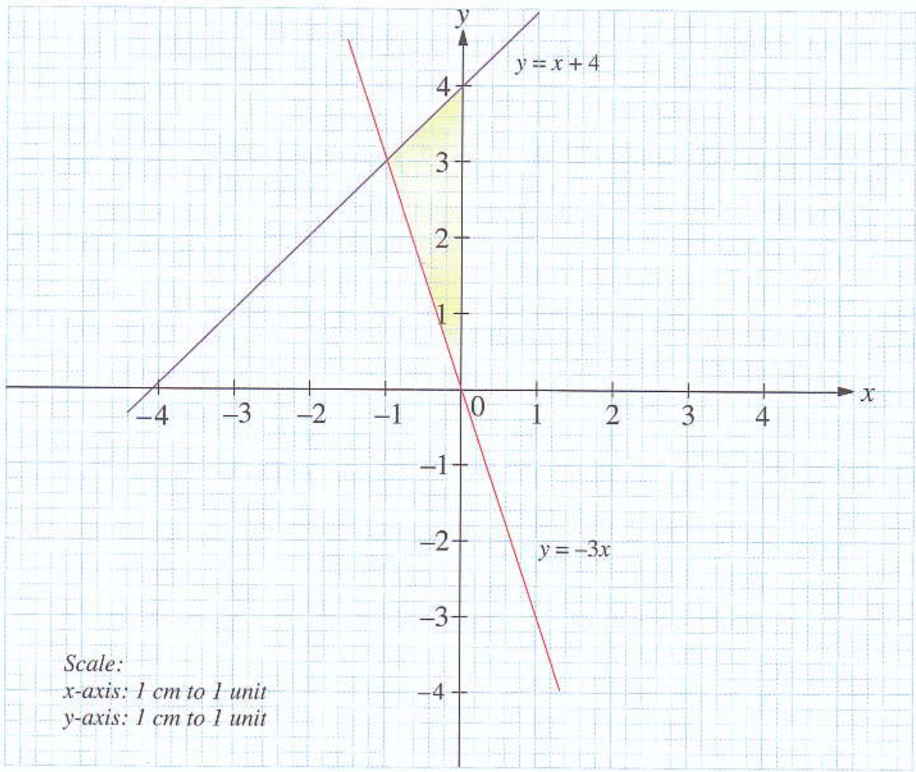
Example 2

Using the same scales and axes, draw the following straight lines:

$$y = x + 4 \text{ and } y = -3x.$$

From the graph, find the coordinates of the point of intersection of the two lines and the area bounded by the two lines and the y-axis.

Solution



The two lines intersect at (–1, 3).

∴ area bounded by the two lines and the y-axis

$$= \frac{1}{2} \times 4 \times 1$$

$$= 2 \text{ units}^2$$

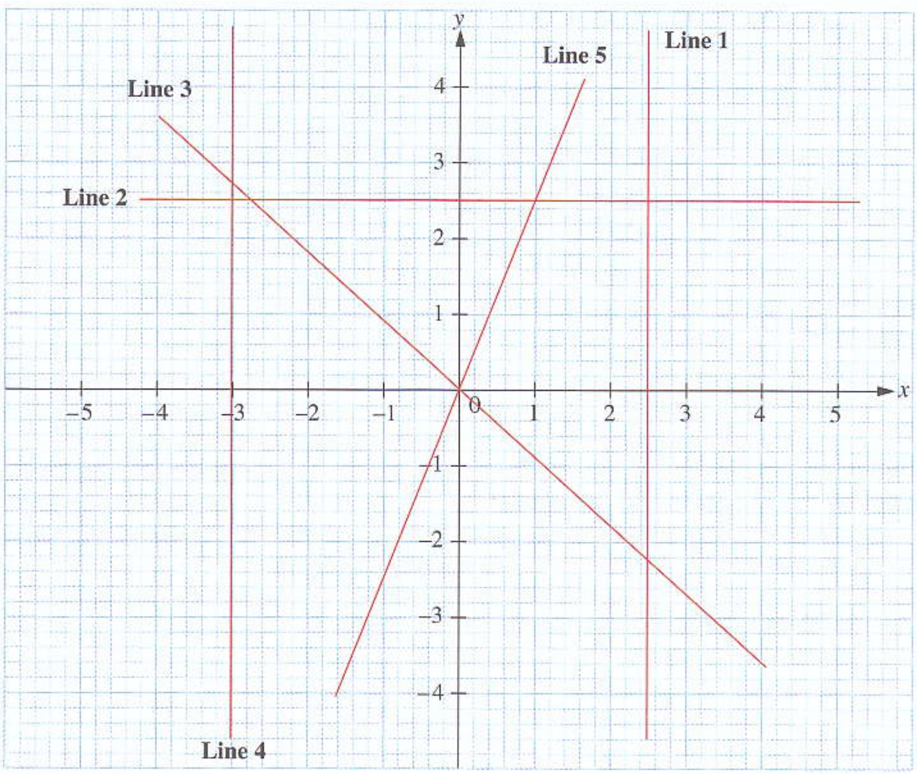
Review Questions 8

1. Using a scale of 2 cm to 1 unit on the x-axis and 1 cm to 2 units on the y-axis for $-5 \leq x \leq 10$, draw the graphs of the following equations:
   (a) $x = -4$ (b) $y = 6$
   (c) $5x - 3y = 45$ (d) $3x + 4y = 8$

2. On separate diagrams, draw the graphs of the following equations and state the gradient of each line:
   (a) $y = 5x$ (b) $2y = 4x + 3$
   (c) $y = 10 - 2x$ (d) $y = 4(x + 1)$

3. (a) Given $y = \frac{1}{4}x + 2$, find the values of y when $x = 0$, 4 and -12. Draw the graph of $y = \frac{1}{4}x + 2$.
   (b) Given $y = 2x + 9$, find the values of y when $x = -1$, $\frac{1}{2}$ and 2. Draw the graph of $y = 2x + 9$.
   (c) Using the graphs of (a) and (b), solve the simultaneous equations $4y = x + 8$ and $y = 2x + 9$.

4. Write down the equations of the lines 1, 2, 3, 4 and 5 in the graph.



5. Draw the graph of each of the following equations on the same graph paper:

(a) $y = -2x + 10$

(b) $y = x + 4$

(c) $y = \frac{1}{4}x + 7$

(d) $y = x - 2$

What figure is formed by these four lines? Write down the coordinates of the vertices of this figure. State the gradient of each line.

6. Using the same scales and axes, draw the following straight lines:

$y = 2$, $y = \frac{2}{3}x$ and $x + y = 8$

(i) From your graph, write down the coordinates of the points of intersection of the three lines.

(ii) Find the area of the triangle bounded by the three lines.

7. Draw the lines of $y = 4$, $x - y = 0$ and $y = -2x$ on the same graph paper.

(i) From your graph, write down the coordinates of points of intersection of the three lines.

(ii) Find the area of the triangle bounded by the three lines.

8. **(a)** Explain why the simultaneous equations $8x - 4y = 20$ and $y = 2x - 33$ have no solution. What can you say about the straight lines representing these two equations?

(b) Explain why the simultaneous equations $y = -\frac{2}{3}x + \frac{4}{3}$ and $12x + 18y = 24$ have an infinite number of solutions. What can you say about the two straight lines representing the equations?

In this chapter, you will learn

- *how to plot quadratic graphs;*
- *the properties of quadratic graphs:*
  - positive or negative coefficient of x^2,
  -maximum and minimum points,
  -symmetry;
- *how to solve problems involving quadratic equations graphically.*

CHAPTER

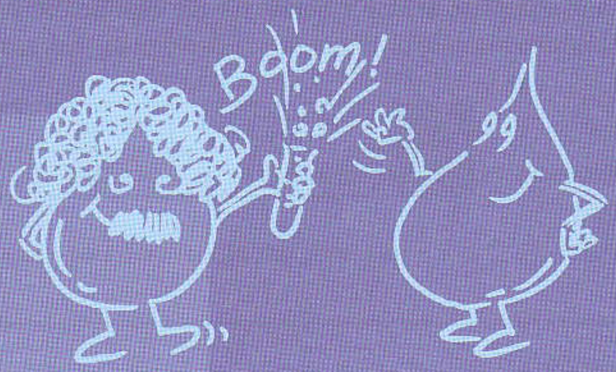
9

Graphs of Quadratic Equations

Introduction

The picture shows the set-up of an experiment to find the relationship between the extension of the length of a spring and the potential energy stored in the spring. From the results of this experiment, a graph can be plotted.

What other experiments can you think of involving two variables?



In Chapter 8, we learnt that the graph of a linear equation in two variables such as $y = 2x + 1$ is a straight line. In this chapter, we shall consider the graph of a quadratic equation in two variables, say, $y = x^2 - x + 1$. We shall see that unlike linear graphs, quadratic graphs involve curves.

Quadratic Equations in Two Variables of the Form $y = ax^2$ $(a \neq 0)$

Explorations

You may work in pairs.

1. The following table of values represents a certain equation in two variables.

x	−4	−3	−2	−1	0	1	2	3	4
y	16	9	4	1	0	1	4	9	16

(a) Examine each ordered pair (x, y) carefully.
- **(i)** What do you notice about the relationship between x and y?
- **(ii)** Write down a formula for the equation, expressing y in terms of x.

(b) **(i)** Construct a pair of axes on a piece of graph paper using 2 cm to represent 1 unit on the x-axis from $x = -5$ to $x = 5$ and 2 cm to represent 5 units on the y-axis from $y = -20$ to $y = 20$.
- **(ii)** Plot the points of the equation using the table given.
- **(iii)** Join the points with a curve.
- **(iv)** Can you describe the shape of the curve?

2. **(a)** Copy and complete the table below for the equation in two variables represented by the graph in Fig. 9.1.

x	−3	−2	−1	0	1	2	3
y							

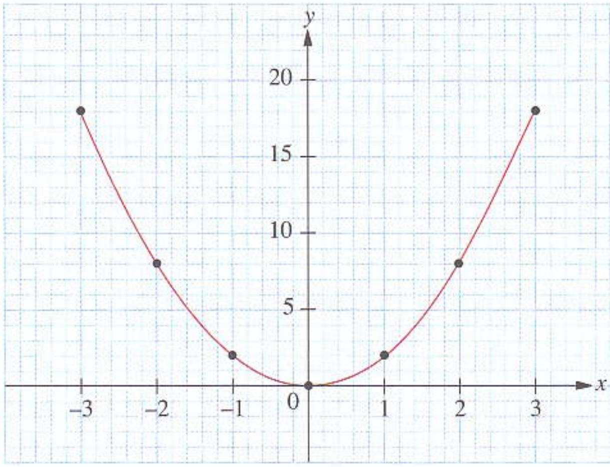


Fig. 9.1

(b) **(i)** Do you notice that for each ordered pair (x, y), the value of y is the result of multiplying the square of the value of x by 2?

(ii) Write down a formula for this equation, expressing y in terms of x.

(c) Plot the graph of this equation on the same pair of axes constructed in **1(b)(i)**.

(d) Compare the two graphs.

3. **(a)** Copy and complete the table of values for the equation in two variables $y = \frac{1}{2}x^2$.

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
y											

(b) Plot the graph of this equation on the same pair of axes constructed in **1(b)(i)**.

(c) Compare the three graphs.

4. How do you expect the graphs of each of the following equations:

$y = 3x^2$, $y = 4x^2$, $y = 5x^2$, $y = \frac{1}{3}x^2$, $y = \frac{1}{4}x^2$, $y = \frac{1}{5}x^2$

to behave in relation to the three graphs you have drawn so far? You may use the computer software, Graphmatica to draw these graphs.

After you have opened **Graphmatica**, go to **view, graph paper** to select **rectangular.** Go to **view** again to select **grid range**.

5. Add the graphs of $y = -x^2$, $y = -2x^2$ and $y = -\frac{1}{2}x^2$ to the same pair of axes constructed in **1(b)(i)**.

From the graphs, what conclusion can you draw about the graphs of $y = ax^2$?

The graphs of $y = ax^2$ $(a \neq 0)$ in the above Exploration can be shown below.

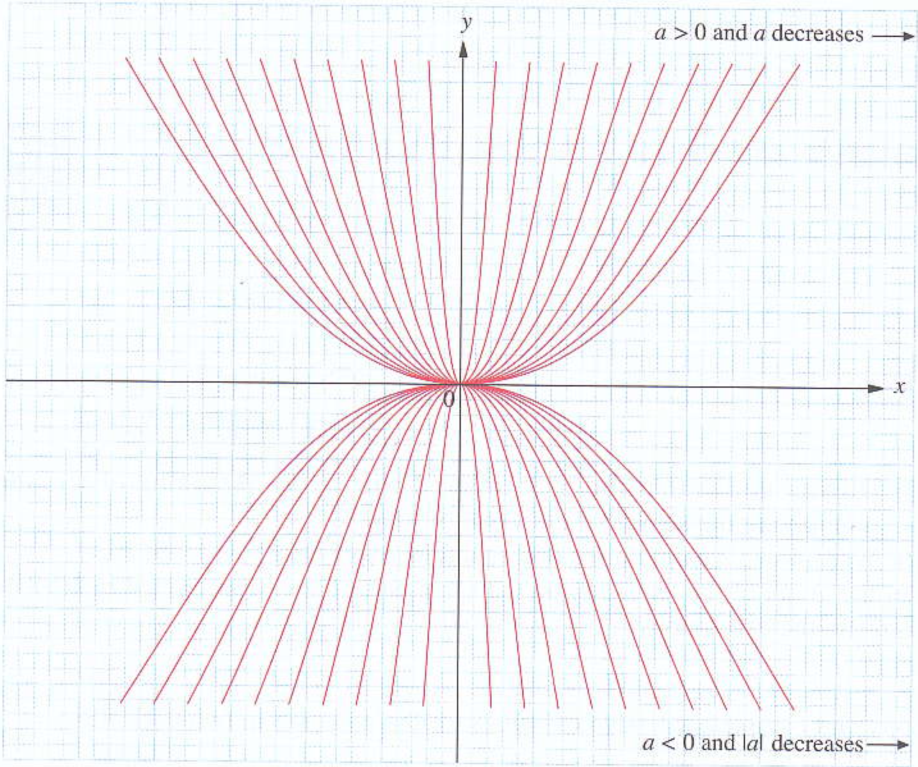


Fig. 9.2

The results from the Exploration can be summarised below.

1. The graphs of $y = ax^2$ $(a \neq 0)$ pass through the origin.

2. The y-axis is the line of symmetry.

3. (a) When a is positive, each graph has a lowest point known as the **minimum point** and opens upwards indefinitely.

(b) The smaller the value of a, the wider the graph opens.

4. (a) When a is negative, each graph has a highest point known as the **maximum point** and opens downwards indefinitely.

(b) The smaller the value of a, the wider the graph opens.

Just FOR fun

Six coins are placed in a straight line with their 'heads' facing upwards. The task is to turn the coins such that you end up having 6 coins with their 'tails' up under the condition that you turn 5 coins in each round. What is the minimum number of rounds required to do this? How many rounds will you need to do this if you have 52 coins instead?

Graphs of General Quadratic Equations in Two Variables

The general form of a quadratic equation is $y = ax^2 + bx + c$ where a, b and c are real numbers and a is not equal to zero.

The steps used for drawing linear graphs in two variables can be used for drawing graphs of any equation in two variables.

Let's revise the steps for drawing graphs in two variables.

(a) Choose a suitable scale for the x-axis and y-axis, so that the graph will be more than half the size of the graph paper.

(b) Construct a table of x and y values for the equation.

(c) Plot the points on the graph paper and join them up to form a smooth curve.

Explorations

We shall now investigate graphs of various quadratic equations in two variables.

Using the above steps for drawing a graph, draw the graphs of

(a) $y = x^2 + 1$
(b) $y = -x^2 + 1$
(c) $y = x^2 + 2x + 2$
(d) $y = 3 + 2x - x^2$.

You have to join up the plotted points for each equation to form a smooth curve.

For each curve, state the following:
(a) the shape of the curve ($\smile$ or $\frown$),
(b) the coefficient of x^2,
(c) the coordinates of the maximum or minimum point,
(d) the equation of the line of symmetry.

Just For Fun

Mr Lin wants to pour 12 litres of water equally into two containers. However, he has only two measuring cans of capacity 9 litres and 5 litres with him. How is he to obtain the two equal amounts of water accurately by using the measuring cans?

Copy the following table and record the information of the curves.

	Quadratic Equation	Shape	Coefficient of x^2	Coordinates of Maximum Point	Coordinates of Minimum Point	Equation of the Line of Symmetry
(a)	$y = x^2 + 1$					
(b)	$y = -x^2 + 1$					
(c)	$y = x^2 + 2x + 2$					
(d)	$y = 3 + 2x - x^2$					

Based on the above information, fill in the blanks.

(a) When the coefficient of x^2 is positive, the graph of the quadratic equation opens _______ and it has a _______ point.

(b) When the coefficient of x^2 is negative, the graph of the quadratic equation opens _______ and it has a _______ point.

(c) The graph of a quadratic equation is symmetrical. The line of symmetry is parallel to the _______ -axis and it passes through the _______ point or the _______ point.

Example 1

Draw the graphs of $y = x^2 + 1$ and $y = -x^2 + 1$ for $-4 \leqslant x \leqslant 4$. Find, from each graph

(a) the value of y when $x = 2.5$,

(b) the coordinates of the maximum or minimum point,

(c) the equation of the line of symmetry.

FOR YOUR Information

The CD, "The Business of Graphs", from the DMS has an interesting section on drawing and reading quadratic graphs. Go through the tutorials and activities. Use the Equation Plotter in the CD to plot a few graphs. You can also use the open tools, Graphmatica or Win-plot, to explore the shapes of quadratic graphs.

Solution

A table of values for both x and y is set up as shown below:

x	-4	-3	-2	-1	0	1	2	3	4
$y = x^2 + 1$	17	10	5	2	1	2	5	10	17
$y = -x^2 + 1$	-15	-8	-3	0	1	0	-3	-8	-15

The graph obtained for each equation is a curve as shown in Fig. 9.3.

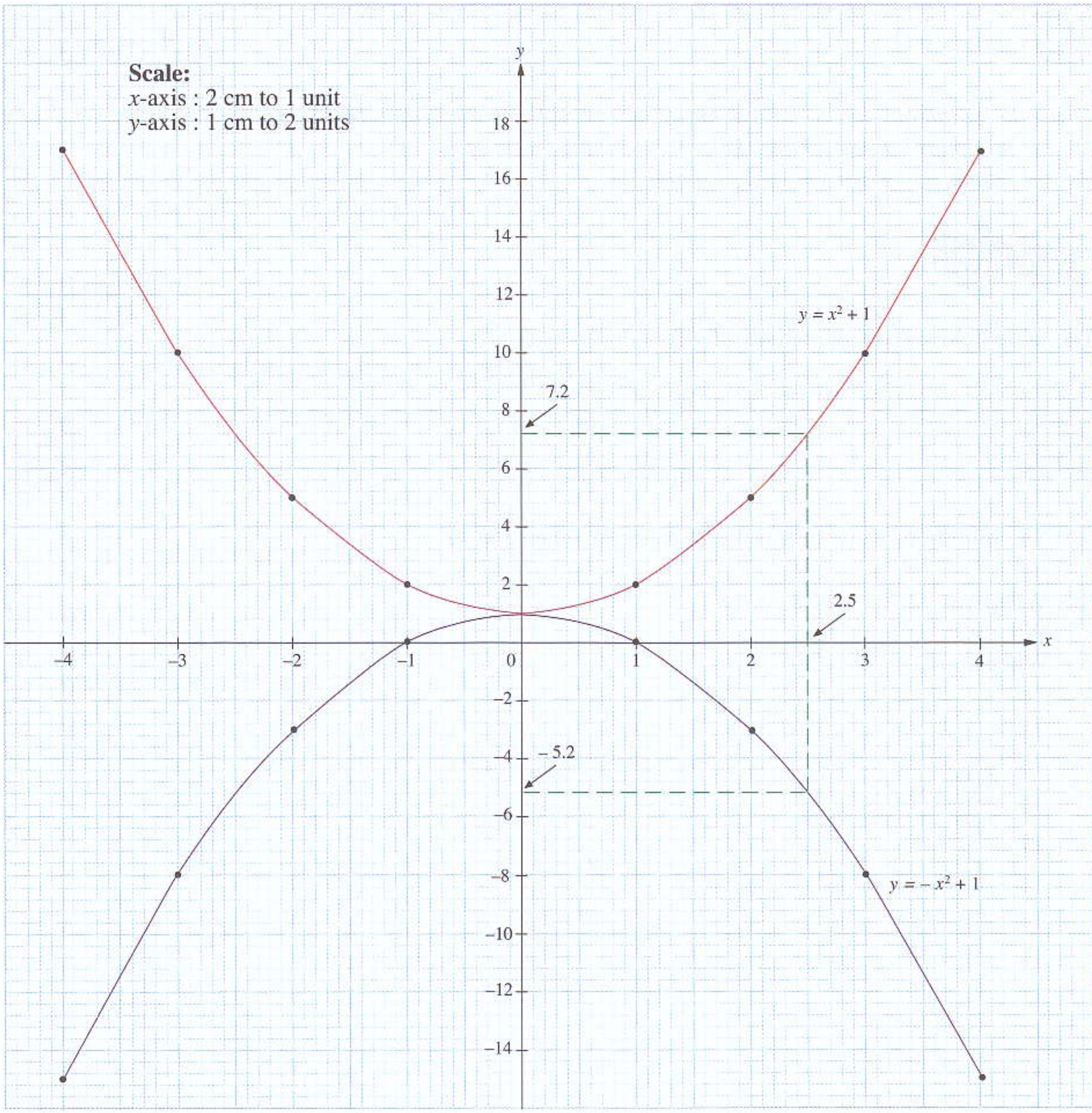


Fig. 9.3

From the graphs,

(a) when $x = 2.5$, y is approximately equal to 7.2 for $y = x^2 + 1$ and y is approximately equal to –5.2 for $y = -x^2 + 1$;

(b) the minimum point of $y = x^2 + 1$ is (0, 1) and the maximum point of $y = -x^2 + 1$ is (0, 1);

(c) for both curves, the y-axis is the line of symmetry and its equation is $x = 0$.

Example 2

Draw the graph of $y = x^2 + 2x + 2$ *for* $-3 \leqslant x \leqslant 3$*. From the graph, find*
(a) the minimum value of y and the corresponding value of x,
(b) the values of y when $x = -2.7$*, 1.4 and 2.3.*

Solution

The table below gives the values of y for $-3 \leqslant x \leqslant 3$.

x	–3	–2	–1	0	1	2	3
y	5	2	1	2	5	10	17

(a) The minimum value of y is 1 and it occurs when $x = -1$.

(b) When $x = -2.7$, $y \approx 4.0$.
When $x = 1.4$, $y \approx 6.8$.
When $x = 2.3$, $y \approx 12.0$.

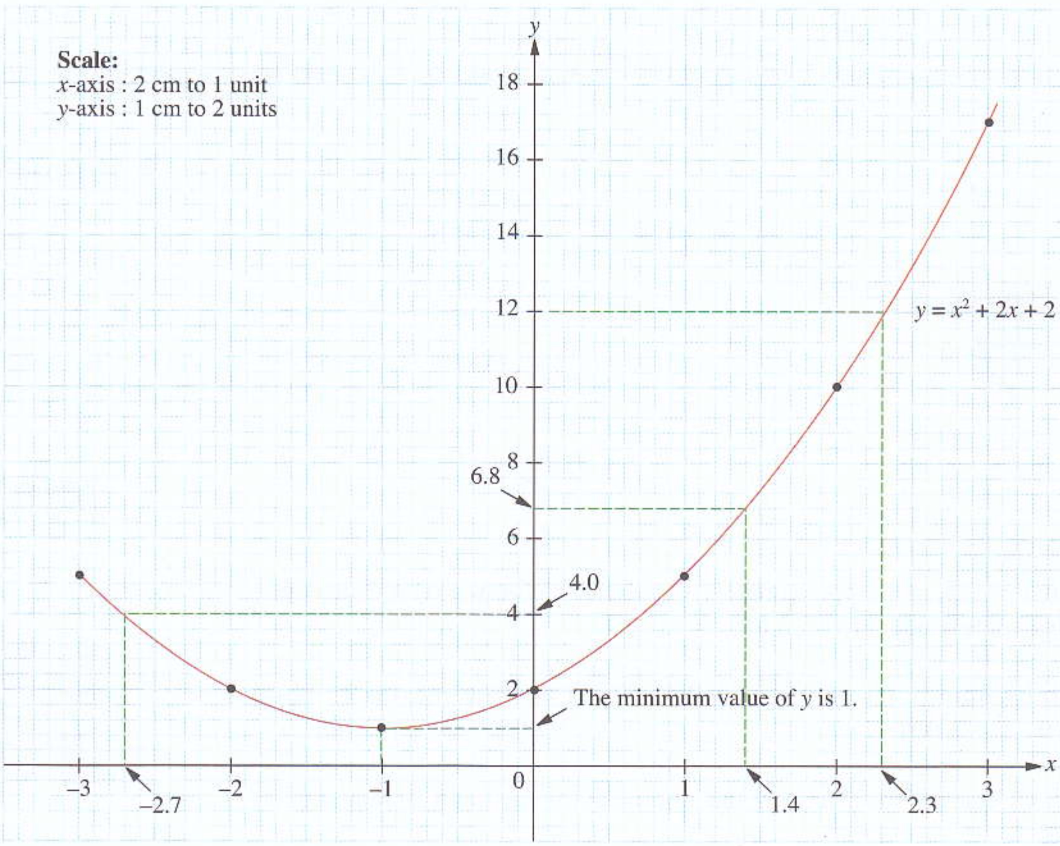


Fig. 9.4

Example 3

Draw the graph of $y = 3 + 2x - x^2$ *for* $-3 \leqslant x \leqslant 5$*. From the graph, find*

(a) the maximum value of y,

(b) the values of y when $x = -1.9$*, 2.7 and 4.3,*

(c) the equation of the line of symmetry of $y = 3 + 2x - x^2$*.*

Solution

The table shows the values of y for $-3 \leqslant x \leqslant 5$.

x	-3	-2	-1	0	1	2	3	4	5
y	-12	-5	0	3	4	3	0	-5	-12

(a) The maximum value of y is 4.

(b) When $x = -1.9$, $y \approx -4.5$.
When $x = 2.7$, $y \approx 1.0$.
When $x = 4.3$, $y \approx -6.8$.

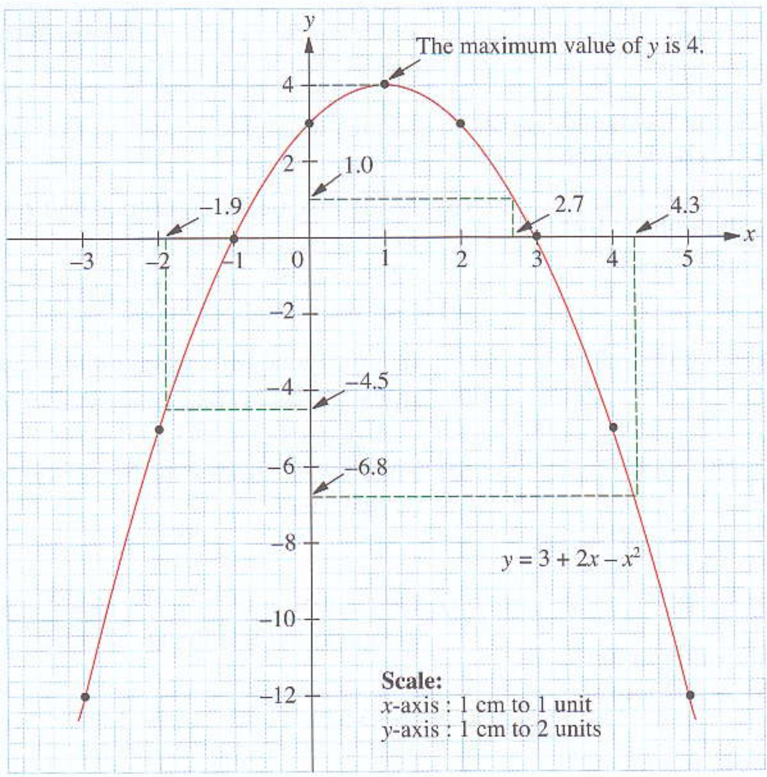


Fig. 9.5

(c) The equation of the line of symmetry is $x = 1$.

Exercise 9a

1. Copy and complete the following table which gives the values of $y = x^2 + 4$ for $-4 \leqslant x \leqslant 4$.

x	-4	-3	-2	-1	0	1	2	3	4
y	20		8					13	

Using 2 cm to represent 1 unit on the x-axis and 1 cm to represent 2 units on the y-axis, draw the graph of $y = x^2 + 4$. Find
(a) the minimum point,
(b) the equation of the line of symmetry.

2. Copy and complete the following table which gives values of $y = 3x - 2x^2$ for $-2 \leqslant x \leqslant 3$.

x	-2	$-1\frac{1}{2}$	-1	0	1	2	$2\frac{1}{2}$	3
y		-9		0		-2		

Using 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = 3x - 2x^2$. Is there a line of symmetry? If so, state the equation of the line of symmetry and find the maximum point.

3. Copy and complete the table which gives values of $y = x^2 - 5x + 4$ for $0 \leqslant x \leqslant 4\frac{1}{2}$.

x	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$
y	4			$-1\frac{1}{4}$	-2			$-1\frac{1}{4}$		$1\frac{3}{4}$

Using 4 cm to represent 1 unit on both axes, draw the graph of $y = x^2 - 5x + 4$. From the graph, find
(a) the minimum value of y and the corresponding value of x,
(b) the equation of the line of symmetry of the graph.

4. **(a)** Given that $y = 2 - 3x - x^2$, copy and complete the following table:

x	-5	-4	-3	-2	-1	0	1	2	3
y	-8			4		2	-2		

(b) Taking 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = 2 - 3x - x^2$ for $-5 \leqslant x \leqslant 3$.
(c) Use your graph to find
  (i) the values of x when $y = -3$, -8 and -13,
  (ii) the values of y when $x = -4.5$, -3.2 and 1.6.

5. **(a)** Given that $y = x^2 - 2x$, copy and complete the following table:

x	-2	-1	0	1	2	3	4
y	8		0	-1			

(b) Using a scale of 2 cm to 1 unit on each axis, draw the graph of $y = x^2 - 2x$ for $-2 \leqslant x \leqslant 4$.
(c) Use your graph to find
  (i) the values of x when $y = -\frac{1}{2}$, 2 and 5,
  (ii) the values of y when $x = -1.8$, 2.2 and 3.5.
(d) Find the equation of the line of symmetry of the curve.

6. **(a)** Given that $y = x^2 + 3x - 4$, copy and complete the following table:

x	-6	-5	-4	-3	-2	-1	0	1	2
y	14							0	

(b) Using 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = x^2 + 3x - 4$ for $-6 \leqslant x \leqslant 2$.

(c) Use your graph to find

(i) the values of x when $y = -2$, 5 and 8,

(ii) the values of y when $x = -5.5$, -2.7 and 1.6,

(iii) the minimum value of y.

7. Construct tables of values for the following equations. Plot their graphs using suitable scales. State the coordinates of the maximum or minimum point.

(a) $y = x^2 - 4x + 3$ $(-4 \leqslant x \leqslant 4)$

(b) $y = 3x^2 + 3x - 5$ $(-2 \leqslant x \leqslant 3)$

(c) $y = -3 + 2x - x^2$ $(-4 \leqslant x \leqslant 4)$

(d) $y = 25 + 4x - 3x^2$ $(-3 \leqslant x \leqslant 5)$

8. Construct the table of values for each equation below, then draw all their graphs on the same pair of axes using suitable scales.

(a) $y = x^2$ $(-3 \leqslant x \leqslant 3)$

(b) $y = x^2 + 3$ $(-3 \leqslant x \leqslant 3)$

(c) $y = x^2 - 3$ $(-3 \leqslant x \leqslant 3)$

How are the graphs related to one another?

9. Construct the table of values for each equation below, then draw all their graphs on the same pair of axes using suitable scales.

(a) $y = -x^2$ $(-3 \leqslant x \leqslant 3)$

(b) $y = -(x - 2)^2$ $(-1 \leqslant x \leqslant 5)$

(c) $y = -(x + 2)^2$ $(-5 \leqslant x \leqslant 1)$

How are the graphs related to one another?

Problems Involving Quadratic Graphs

Many problems lead to quadratic equations. We shall now look at some problems involving quadratic graphs.

Example 4

A triangle has base $(x + 2)$ cm and height $(7 - x)$ cm.

(a) Write down an equation for the area, A, of the triangle.

(b) By constructing a table of values for $-3 \leqslant x \leqslant 8$, plot the graph of A against x and find the base and height that will result in the maximum area of the triangle.

Solution

(a) $A = \frac{1}{2}(x + 2)(7 - x)$

$= \frac{1}{2}(14 + 5x - x^2)$

$= 7 + \frac{5}{2}x - \frac{x^2}{2}$

(b)

x	−3	−2	−1	0	1	2	3	4	5	6	7	8
A	−5	0	4	7	9	10	10	9	7	4	0	−5

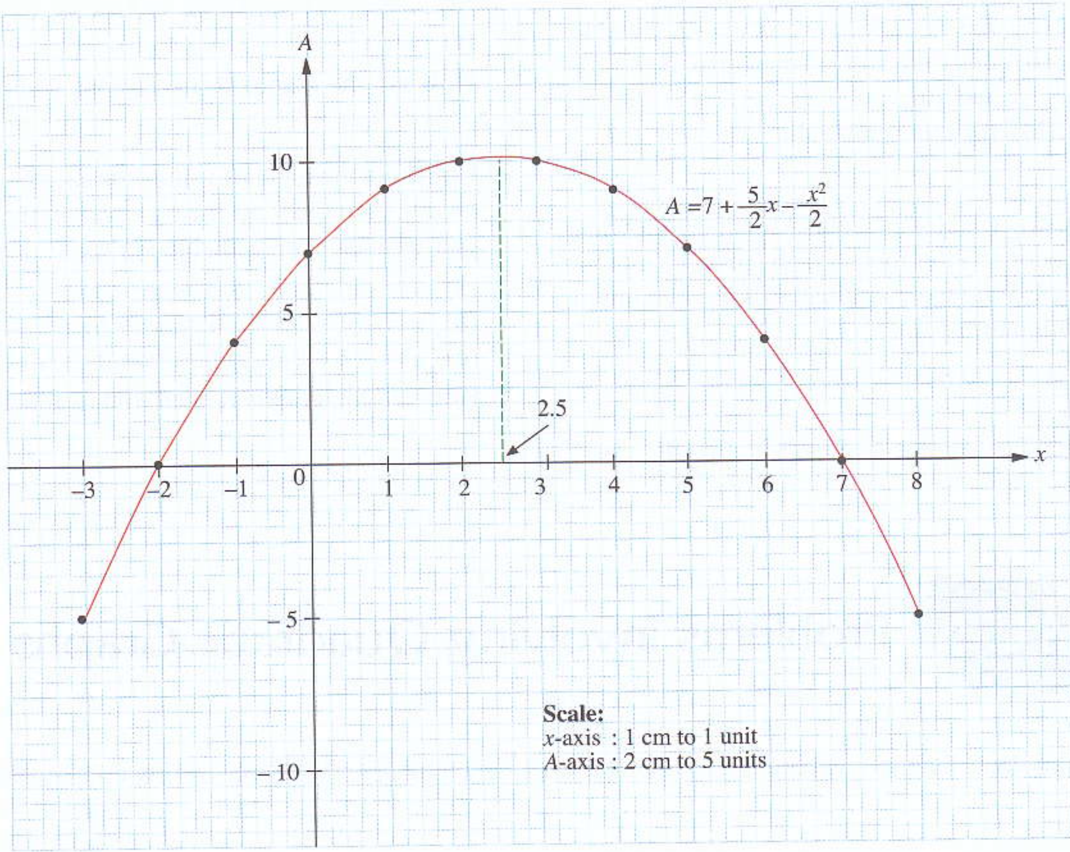


Fig. 9.6

From the graph, the value of x that corresponds to the maximum value of A is 2.5.
Therefore, the corresponding base and height are (2.5 + 2) cm and (7 – 2.5) cm, i.e. 4.5 cm and 4.5 cm.

Example 5

Peter makes and sells handmade toys. He finds that if a batch of x toys is made, where $1 \leqslant x \leqslant 14$, the cost per toy \$y is given by $y = x^2 - 18x + 110$.

(a) Draw the graph of $y = x^2 - 18x + 110$ from $x = 0$ to $x = 14$, using 1 cm to represent 1 unit on the x-axis and 1 cm to represent 20 units on the y-axis.

(b) Use the graph to write down the number of toys in a batch such that the cost per toy is

(i) a minimum,

(ii) less than \$60.

Solution

The table below shows the values of y for values of x from 0 to 14.

x	0	1	2	4	6	8	10	12	14
y	110	93	78	54	38	30	30	38	54

(a) The graph of $y = x^2 - 18x + 110$ is drawn as shown below.

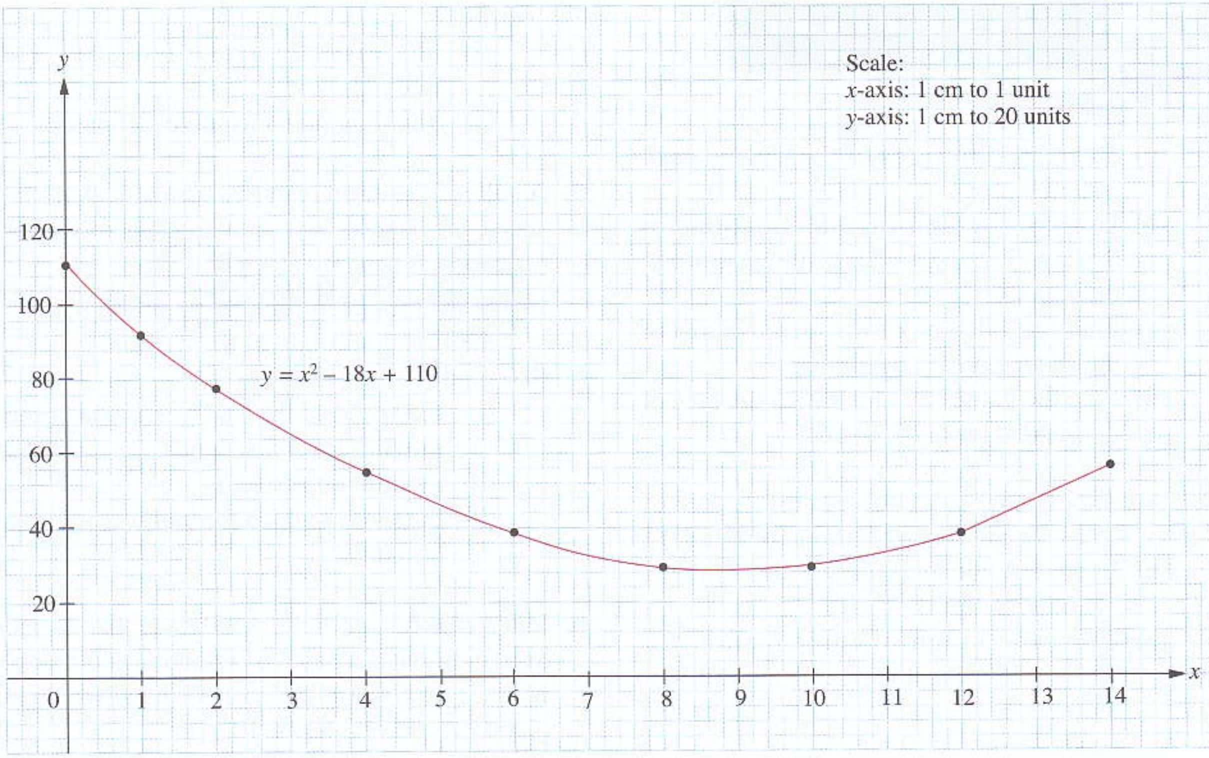


Fig. 9.7

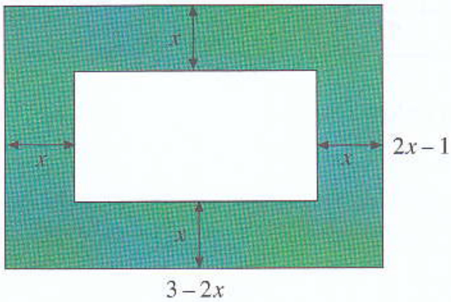
(b) From the graph,

(i) y is minimum when $x = 9$.
Thus, the number of toys in a batch that will result in the minimum cost per toy is 9.

(ii) $y < 60$ when $x \geqslant 4$.
Thus, the number of toys in a batch that will result in the cost per toy being less than \$60 are 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14.

Exercise 9b

1. The figure below shows two rectangles and the length (in m) of their sides expressed in terms of x.



(a) Express A, the area of the shaded part, in terms of x.

(b) Draw the graph of the equation for $0 \leqslant x \leqslant 1$, using a scale of 10 cm to 1 unit on both axes.

(c) Find the value of x for which the area of the shaded part is maximum.

(d) Find the values of x for which the area of the shaded part is $0.72\ \text{m}^2$.

2. A solid cylinder has a curved surface area of $(x^2 - 3x + 2)\ \text{cm}^2$ and a base area of $x^2\ \text{cm}^2$.

(a) Express A, the total surface area of the cylinder, in terms of x.

(b) Draw the graph of the equation for $0 \leqslant x \leqslant 2$, using a scale of 5 cm to 1 unit on the x-axis and 2 cm to 1 unit on the y-axis.

(c) Find the minimum total surface area.

(d) Find the value of x for which the total surface area is $5\ \text{cm}^2$.

3. If a store prices each item at $\$x$, $(64 - 8x)$ items are sold.

(a) The total amount of money received for selling these items is $\$y$. Express y in terms of x.

(b) Draw the graph of the equation for $0 \leqslant x \leqslant 8$, using a scale of 2 cm to 1 unit on the x-axis and 1 cm to 10 units on the y-axis.

(c) Find the price at which the store should charge each item so that the total amount received is maximum.

4. An object sliding down a slope has travelled a distance, s metres, in time, t seconds, where $s = 4t + t^2$.

(a) Draw a graph to show the distances covered up to 5 seconds.

(b) Find

(i) the distance travelled after 2.6 seconds,

(ii) the time taken to travel 30 metres.

5. A ball rolling on an uneven slope with an initial speed of 10 m/s, was moving at v m/s after t seconds, where $v = 2t^2 - 8t + 10$.

(a) Draw the speed-time graph of the ball for the first 5 seconds of the motion.

(b) Find

(i) the speed of the ball when it has been moving for 3.8 seconds,

(ii) its minimum speed,

(iii) the time at which the ball was moving at 6 m/s.

6. Mary makes and sells handmade handbags. She finds that if a batch of x handbags is made, where $1 \leqslant x \leqslant 14$, the cost per handbag $\$y$ is given by $y = x^2 - 16x + 100$.

(a) Draw the graph of $y = x^2 - 16x + 100$ for $0 \leqslant x \leqslant 14$, using 1 cm to represent 1 unit on the x-axis and 2 cm to represent 10 units on the y-axis.

(b) Use the graph to write down the number of handbags in a batch that will make the cost per handbag

(i) a minimum,

(ii) less than \$70.

SUMMARY

1. The general form of a quadratic graph is $y = ax^2 + bx + c$ $(a \neq 0)$.

2. The quadratic graph of $y = ax^2 + bx + c$ $(a \neq 0)$ has a minimum point (the lowest point) when a is positive. It has a maximum point (the highest point) when a is negative.

3. The line of symmetry of the quadratic graph passes through the maximum or minimum point.

Review Examples 9

Example 1

Fandi kicks a soccer ball vertically upwards. The height, h metres, of the ball after t seconds is given by

$$h = 27t - 6t^2.$$

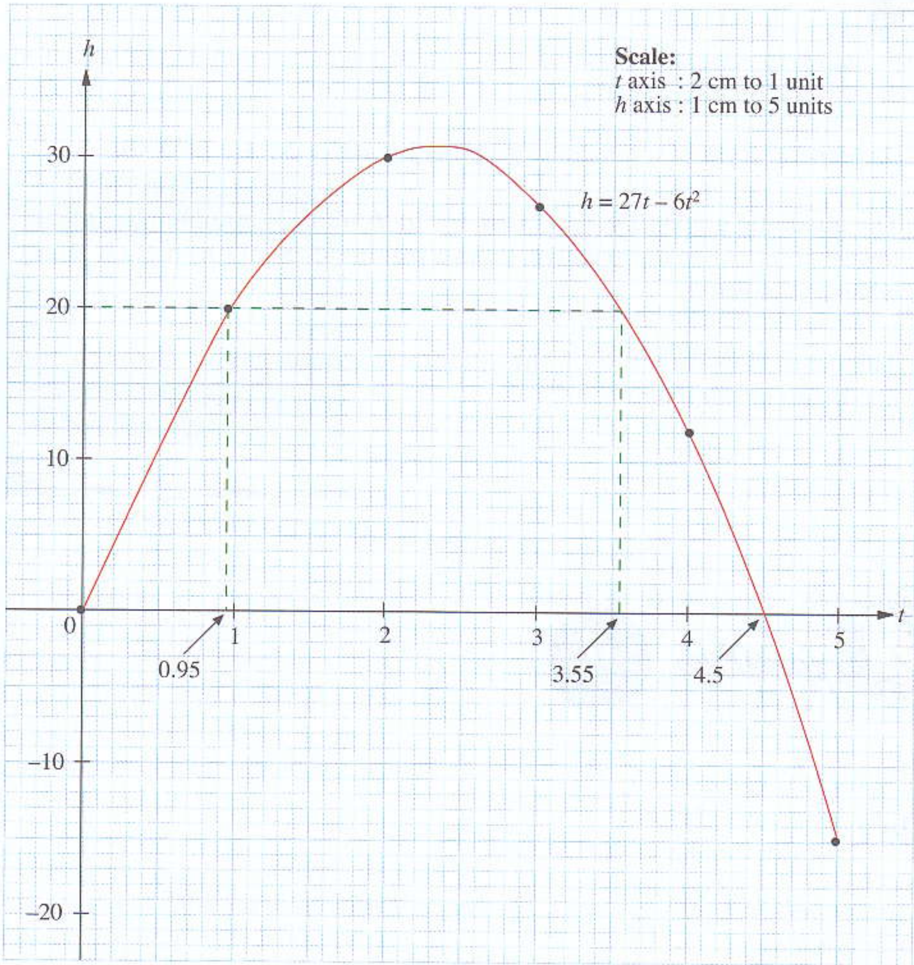
By drawing a suitable graph, find

(a) the maximum height of the ball above the ground,
(b) the time required for the ball to reach the ground again,
(c) the shortest time taken to reach a height of 20 metres.

Solution

The table below shows the values of h for values of t from 0 to 5.

t	0	1	2	3	4	5
h	0	21	30	27	12	–15



From the graph,

(a) the maximum h is 30.5 m;

(b) when $h = 0$, $t = 0$ or 4.5.
The time taken for the ball to reach the ground again is 4.5 s.

(c) when $h = 20$ m, $t = 0.95$ s or 3.55 s.
The shortest time taken for the ball to reach a height of 20 m is 0.95 s.

Example 2

In the sketch, the curve $y = x^2 + 2x - 3$ cuts the x-axis at two points A and B, and the y-axis at point C. M is the minimum point.
Write down the coordinates of points A, B, C and M.

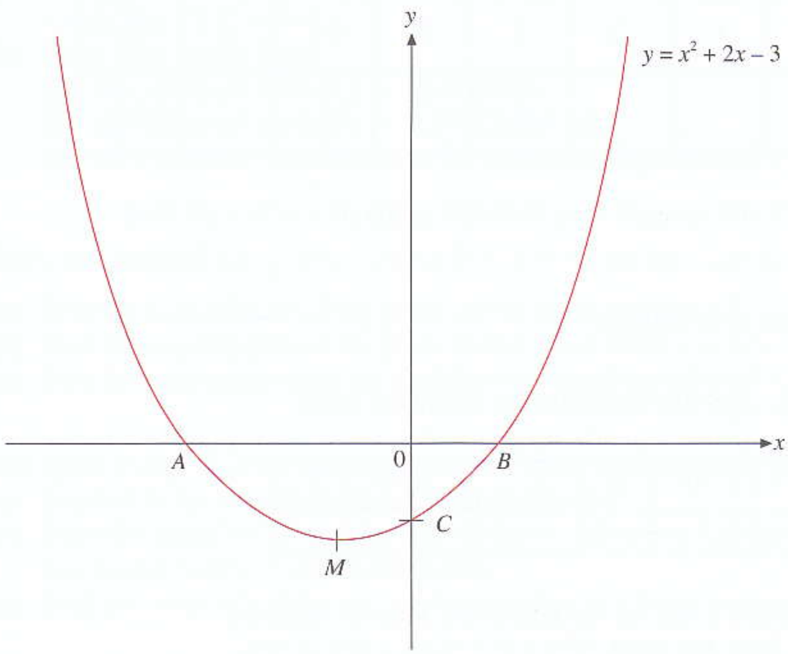


Fig. 9.8

Solution

$y = x^2 + 2x - 3 = (x - 1)(x + 3)$
When $y = 0$, $(x - 1)(x + 3) = 0$
$x = 1$ or -3

Hence the coordinates of A are $(-3, 0)$ and B are $(1, 0)$.
To find C, let $x = 0$ and we have $y = -3$
$\therefore$ the coordinates of C are $(0, -3)$

To find M, we have to find the x value of the mid-point of A and B,

i.e. $x = \dfrac{-3+1}{2} = -1$

Substitute $x = -1$ into $y = x^2 + 2x - 3$, we have $y = -4$.

Hence the coordinates of M are $(-1, -4)$.

Review Questions 9

1. Copy and complete the following table which gives values of $y = 5x - x^2$ for $-\frac{1}{2} \leqslant x \leqslant 5\frac{1}{2}$.

x	$-\frac{1}{2}$	0	1	2	3	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$
y	$-2\frac{3}{4}$	0			6		$2\frac{1}{4}$		$-2\frac{3}{4}$

Using 2 cm to represent 1 unit on each axis, draw the graph of $y = 5x - x^2$. Find

(a) the maximum point,
(b) the equation of the line of symmetry.

2. **(a)** Given that $y = x^2 - 4$, copy and complete the following table:

x	-3	-2	-1	0	1	2	3	4	5
y		0	-3		-3		5		

(b) Taking 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = x^2 - 4$ for $-3 \leqslant x \leqslant 5$.
(c) From your graph, find
(i) the values of y when $x = -0.5$, 1.5 and 3.5,
(ii) the values of x when $y = -2$, 6 and 8,
(iii) the minimum point.

3. **(a)** Given that $y = x^2 - 2x + 1$, copy and complete the following table:

x	-4	-3	-2	-1	0	1	2	3	4
y	25			4			1	4	

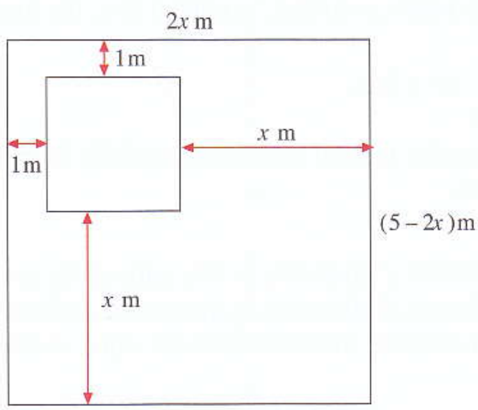
(b) Taking 2 cm to represent 1 unit on the x-axis and 1 cm to represent 2 units on the y-axis, draw the graph of $y = x^2 - 2x + 1$ for $-4 \leqslant x \leqslant 4$.
(c) From your graph, find
(i) the values of x when $y = 3$, 8 and 14,
(ii) the values of y when $x = -2.4$, 0.2 and 3.7,
(iii) the minimum point,
(iv) the equation of the line of symmetry.

4. **(a)** Taking 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = 10 - x - x^2$ for values of x such that $-4 \leqslant x \leqslant 3$.
   (b) Use your graph to find the values of
   (i) x when $y = -1$, 5 and 8,
   (ii) y when $x = -2.2$, 1.6 and 2.5.

5. **(a)** Using 2 cm to represent 1 unit on each axis, draw the graph of $y = x^2 - 2x - 5$ for values of x such that $-2 \leqslant x \leqslant 4$.
   (b) From your graph, find
   (i) the values of x when $y = -3$, 0 and 2,
   (ii) the values of y when $x = -1.5$, 0.2 and 2.6,
   (iii) the minimum value of y and the corresponding value of x.

6. Draw the graph of $y = \frac{1}{5}(13x - x^2)$ for $0 \leqslant x \leqslant 14$, taking 1 cm to represent 1 unit on both axes.
   (a) Find the x-coordinate of the point on the curve when $y = 5$.
   (b) Find the y-coordinate of the point on the curve when $x = 6.5$.
   (c) Find the maximum value of y and the corresponding value of x.

7. The figure below shows a rectangular cement floor surrounded by grassland.
   (a) Express A, the area of the grassland in terms of x.
   (b) Draw the graph of the equation for values of x from 0 to 4, using a scale of 4 cm to 1 unit on the x-axis and 2 cm to 1 unit on the y-axis.
   (c) Find the value of x if the area of the grassland is to be as large as possible.



8. A bookshop finds that if it prices each magazine at $x, (85 – 6x) copies will be sold.

(a) The total amount received for selling these magazines is $y. Express y in terms of x.

(b) Using 1 cm to 1 unit on the x-axis and 1 cm to 20 units on the y-axis, draw the graph of the equation for $0 \leqslant x \leqslant 12$.

(c) Find the price at which the bookshop should charge each magazine if the total amount received is to be as large as possible.

(d) If the bookshop charges the magazine at $4.50 each, find the total amount received.

9. A factory finds that its daily profit, y dollars, is related to x, the number of items it produces daily where $y = -x^2 + 90x$.

(a) Draw the graph of $y = -x^2 + 90x$.

(b) Use your graph to find

(i) the number of items the factory must produce daily in order to maximize profit,

(ii) the maximum profit.

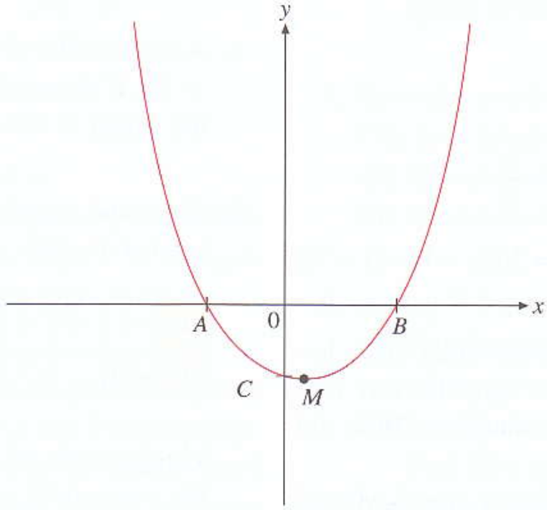
10. Tobey sits by the lake and throws a stone into it. The path of the stone is described by the equation $h = 22t - 5t^2$ where h is the height of the stone in metres at t seconds after it has been thrown. By drawing a suitable graph, find the time taken for the stone to reach

(a) the maximum height,

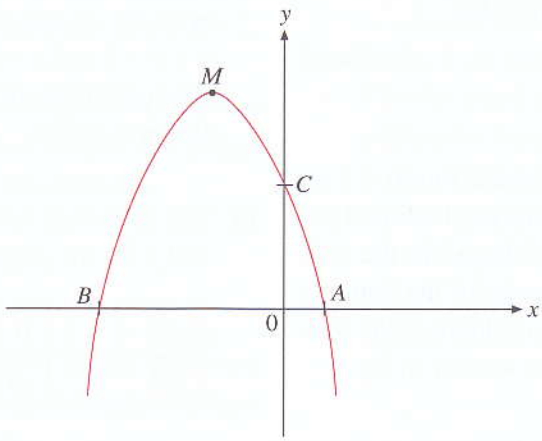
(b) the surface of the lake.

The stone takes 5.5 seconds to reach the bottom of the lake. Assuming the stone travels along the same path in the water, find the depth of the lake.

11. In the sketch, the curve $y = x^2 - 2x - 8$ cuts the x-axis at two points A and B, and the y-axis at point C. Given that M is the minimum point. Write down the coordinates of the points A, B, C and M.



12. In the sketch, the curve $y = (1 - x)(x + 5)$ cuts the x-axis at two points A and B, and the y-axis at C. Given that M is the maximum point, write down the coordinates of the points A, B, C and M.



Revision Exercise III No. 1

1. Solve the following equations:

   (a) $2(x - 3) = 8 - 3(x - 2)$

   (b) $\frac{1}{3}x = \frac{1}{7}(90 - x)$

   (c) $\frac{2(x-1)}{3} - \frac{5(x-3)}{6} = \frac{3}{4}$

   (d) $3[(x - 2) - (2x - 1)] = 2[(2x + 1) - (x + 2)]$

2. (a) A mother is 30 years older than her daughter. Five years ago, she was four times as old as her daughter. How old are they now?

   (b) Find two numbers whose sum is 90 and one-third of the smaller number is equal to one-seventh of the larger number.

3. A regular pyramid stands on a square base of sides 8.6 cm each. If the height of the pyramid is 9.2 cm, calculate

   (a) the volume of the pyramid,

   (b) the length of a sloping edge.

   Give your answers correct to 4 significant figures.

4. A rectangular tank has a base 2.4 m by 1.8 m. It is being filled with 230 litres of a liquid per minute. Find the depth of liquid in the tank after 8 minutes. If the density of the liquid is 1.25 g/cm^3, find the mass of liquid in the tank after 8 minutes. Give your answer in kg.

5. (a) If $N = L(1 - d)$, find d in terms of N and L. Given that $N = 40$ and $L = 50$, find the value of d.

   (b) If the angles of a pentagon are $x°$, $1\frac{1}{2}x°$, $2\frac{1}{2}x°$, $3x°$ and $(2x - 20)°$, find the value of x.

6. In ΔABC, $\angle B = 90°$, $AB = 12$ cm and $AC = 13$ cm. Find

   (a) the length of BC,

   (b) the area of ΔABC.

7. A scale model of a ship is made to a scale of 1 : 80. If the mast of the model is 42 cm, find the height of the actual mast.

8. Copy and complete the following table for $y = 6 + x - 2x^2$.

x	–3	–2	–1	0	1	2	3
y	–15		3		5		–9

   Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = 6 + x - 2x^2$ for $-3 \leqslant x \leqslant 3$.

   Use your graph to estimate

   (a) the value of y when $x = 0.4$,

   (b) the values of x when $y = -6$.

9. Using a scale of 2 cm to represent 1 unit on both the x- and y-axes, draw the graphs of $y = x - 1$ and $x + y = 5$ and hence determine the coordinates of the point of intersection of the two graphs.

10. The following table shows some values of x and y for the graph $y = x^2 - 2x - 3$.

x	–2	–1	0	1	2	3	4	5
y			–3	–4		0	5	

   Copy and complete the table above and draw the graph of $y = x^2 - 2x - 3$ for $-2 \leqslant x \leqslant 5$. Draw the line of symmetry on your graph and state its equation. State the solution of the equation $x^2 - 2x - 3 = 0$.

Revision Exercise III No. 2

1. Solve the following equations:
   (a) $3x - 2(3x - 1) + 4(x + 1) = 0$
   (b) $6 - [(3x - 7) - (7x - 3)] = 0$
   (c) $4(3x - 2) = 7x - 3(2x - 1)$
   (d) $6x - [7x - (8x - 19)] = 2$

2. (a) If $7 : 15 = x : 3$, find x.
   (b) A sum of money is divided between two boys, A and B, in the ratio 5 : 3. If B gets \$9, how much will A get?

3. (a) The line $2x + 3y = 12$ cuts the x- and y-axes at the points A and B respectively. Find the coordinates of the points A and B by plotting the graph $2x + 3y = 12$ on a coordinate graph. If C is the point $(0, -2)$, find the area of the triangle ABC.
   (b) The coordinates of the point of intersection of the lines $ax + y = 3$ and $x + 2y = b$ are $(2, -3)$. Find a and b.

4. (a) Given that $v^2 = u^2 + 2as$, express s in terms of u, v and a. Find the value of s if $u = 2$, $v = 7$ and $a = 4\frac{1}{2}$.
   (b) The sum of two numbers is 94. If twice the smaller number minus the larger number is 26, find the numbers.

5. Factorise $20a^3 - 45a$.

6. (a) A pyramid stands on a square base with sides of 9 cm each. If its height is 15 cm, calculate its volume.
   (b) Find the diameter of a circular cylinder of volume 100 cm^3 and height 6.8 cm. Give your answer correct to 3 significant figures.

7. (a) Find the volume and total surface area of a closed cylinder of height 20 cm and radius 10 cm.
   (b) Find the volume and surface area of a sphere of radius 6.5 cm,

8. Draw the graph of $y = x^2 + 2$ for $-3 \leqslant x \leqslant 3$. Use your graph to find
   (a) the value of y when $x = 1.7$,
   (b) the values of x when $y = 7.5$.

9. A map is drawn to a scale of 4 cm to 3 km.
   (a) Find the scale of the map in the form $1 : n$ where n is an integer.
   (b) Two places on the map are 23 cm apart. Find their actual distance.
   (c) A forest reserve has an area of 24 cm^2 on the map. Find its actual area.

10. Copy and complete the following table of values for $y = 3 + 2x - x^2$.

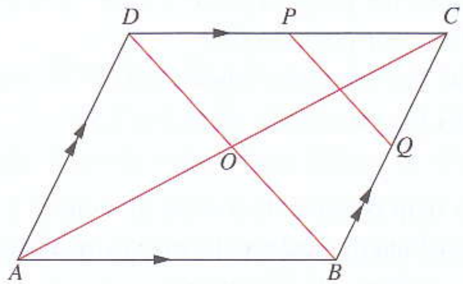
x	−3	−2	−1	0	1	2	3	4
y	−12			3	4			−5

Plot the graph of $y = 3 + 2x - x^2$ using a scale of 2 cm to 1 unit on the x-axis and 2 cm to 20 units on the y-axis. Use your graph to find
   (a) the greatest value of y,
   (b) the equation of the line of symmetry,
   (c) the values of x when $3 + 2x - x^2 = 0$,
   (d) the values of x when $3 + 2x - x^2 = -2$.

Revision Exercise III No. 3

1. Make c the subject of the formula $m = \frac{b + 2c}{3}$.

2.



$ABCD$ is a parallelogram whose diagonals intersect at O. P and Q are the midpoints of DC and BC respectively.

(a) Name a pair of congruent triangles and state briefly how one triangle may be transformed to the other.

(b) Name a pair of similar triangles and state briefly how the smaller triangle may be transformed to the bigger triangle.

3. Solve the simultaneous equations

$$2x + y = 1,$$
$$4x + 5y = 8.$$

4. Simplify each of the following:

(a) $\frac{x}{5} + \frac{2x - 1}{3}$ (b) $\frac{1}{x} - \frac{2}{x + 2}$

5. (a) A polygon has n sides. Four of its exterior angles are 12°, 25°, 32° and 41° and the remaining $(n - 4)$ exterior angles are each equal to 50°. Find n.

(b) The Singapore government gave out $35 million to 121 000 students in the form of Edusave scholarships and bursaries in 1999. Calculate the average amount received by each student giving your answer correct to the nearest 50 cents.

6. (a) Find the volume of a cylindrical steel bar 20 cm long and 3.2 cm in diameter. Find also its total surface area. Give your answer correct to 1 decimal place. (Take $\pi = 3.14$.)

(b) A rectangular tank 50 cm long and 40 cm wide contains 100 litres of water. Find the depth of the water.

7. Factorise the following:

(a) $36x^2 - 49y^2$

(b) $12a^2 - 31a - 15$

*(c) $h^2 - 3hk - 54k^2$

8. $ABCD$ is a rectangle in which $AB = 16$ cm, $AC = 20$ cm and AE is the perpendicular from A to BD. Find the lengths of AD and AE.

9. Copy and complete the following table for $y = 2x^2 - 9x + 2$.

x	0	1	2	3	4	5	6	7
y	2		–8	–7		7	20	

Using a scale of 2 cm to represent 1 unit on the x-axis and 2 cm to represent 5 units on the y-axis, draw the graph of $y = 2x^2 - 9x + 2$ for $0 \leqslant x \leqslant 7$.

Use your graph to estimate

(a) the value of y when $x = 4.6$,

(b) the value of x when $y = -5$.

10. (a) A rectangular solid 24 cm by 22 cm by 21 cm is melted down and recast into a solid cone of base radius 28 cm. Find the height of the cone.

(b) Find the surface area of a sphere of diameter 28 cm.

(c) A right pyramid has a rectangular base 14 cm long and 8 cm wide. Find the height of the pyramid if the volume is 336 cm^3.

Revision Exercise III No. 4

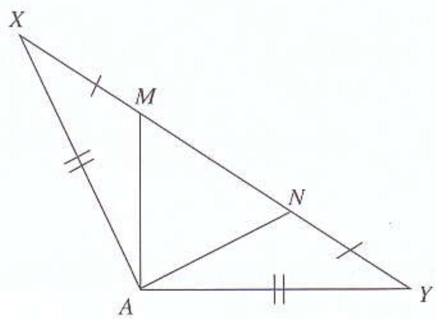
1. (a) If $4 : x = 9 : 14$, find x.
   (b) A number is divided into two parts in the ratio $7 : 4$. If the larger part is 21, find the number.

2. Factorise $a^2 + 2ab + b^2 - c^2$.

3. Simplify (a) $\frac{x}{3} - \frac{2x}{7}$,
   (b) $\frac{x+1}{x-2} - \frac{1}{x}$.

4. Solve the simultaneous equations
   $4x - 5y = 15$,
   $\frac{x+y}{3} - \frac{x-y}{4} = 1$.

5. (a)

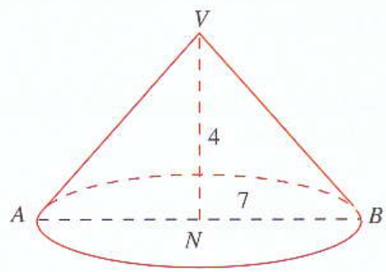


   In the figure shown above, $AX = AY$, $MX = NY$ and $XMNY$ is a straight line. Show that $A\hat{M}N = A\hat{N}M$.
   (b) A rhombus of sides 8 cm each has a diagonal 13 cm long. Find the length of its other diagonal.

6. The area of one face of a cube is 25 cm^2. Find
   (a) the volume of the cube,
   (b) the total length of all the edges of the cube.

7. (a) The line $3y = 5x + 17$ has the same gradient as the line $5y - kx - 19 = 0$. Find the value of k and hence state the coordinates of the point where the line $5y - kx + 3k = 0$ cuts the y-axis.
   (b) Choose a suitable scale to draw the graphs of $y = 2x - 8$ and $2y + 3x = 5$. Write down the point of intersection of the two graphs.

8. In a right circular cone, the base diameter $AB = 7$ cm and the height VN of the cone is 4 cm. Calculate
   (a) the volume,
   (b) the slant side VA of the cone.



9. (a) A rectangular tank 3 m long and 2.4 m wide internally contains 7200 litres of liquid. Find the depth of the liquid in the tank.
   (b) A cylindrical drum of diameter 24.6 cm can hold 2.46 litres of water. Find the height of the water level, giving your answer correct to the nearest centimetre.

10. Copy and complete the following table of values for $y = 3x - x^2$.

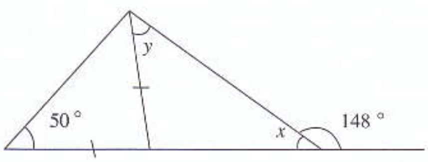
x	−2	−1	0	1	2	3	4
y	−10		0		2		−4

Using a scale of 2 cm to 1 unit on both axes, plot the graph of $y = 3x - x^2$ for $-2 \leqslant x \leqslant 4$. Use your graph to answer the following:
   (a) What is the greatest value of $3x - x^2$?
   (b) What is the value of y when $x = 2.5$?
   (c) What are the possible values of x when $y = -2$?

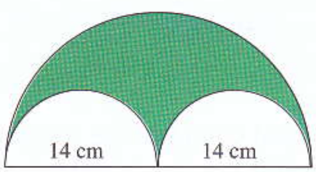
Revision Exercise III No. 5

1. (a) The volume of a cone is 78.8 cm^3 and its base radius is 4.2 cm. Taking π to be 3.142, find the height of the cone. Give your answer correct to 3 significant figures.
   (b) A cylindrical metal bar of length 2 m and diameter 2 cm is melted to form a circular disc of thickness 1 cm. Find the diameter of the disc.

2. (a) Find the angles marked x and y in the figure below.



   (b) The figure below shows two identical semicircles inside a larger one. Find the area and perimeter of the shaded portion of the figure. (Take π to be 3.142.)



3. Solve the simultaneous equations

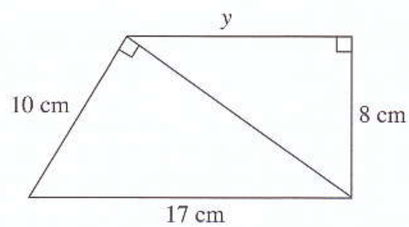
$$3(x-2)+\frac{y+5}{2}=2,$$
$$\frac{x+y}{2}+3x=7.$$

4. Given that $x^2+y^2=15$ and $xy=2$, find the value of $(x-y)^2$.

5. A wire is in the form of a circle of radius 14 cm. If the wire is bent into the shape of a square, find
   (a) the perimeter of the square,
   (b) the area of the square.
   (Take π to be 3.142.)

6. Solve the following equations:
   (a) $3x^3-10x^2-8x=0$
   (b) $(2x-4)(3x+1)=12$

7. (a) A ladder 5 m long leans against a vertical wall. Its foot is 1 m away from the base of the wall. What height does the ladder reach?
   (b) Calculate the length y in the figure below.



8. Copy and complete the following table for $y=x^2-2x-4$.

x	–3	–2	–1	0	1	2	3	4	5
y		4	–1	– 4			–1		11

Using a scale of 2 cm to represent 1 unit on the x-axis, and 2 cm to represent 2 units on the y-axis, plot the graph of $y=x^2-2x-4$ for $-3 \leqslant x \leqslant 5$. Use your graph to answer the following questions:
   (a) What is the value of y when $x = 1.4$?
   (b) What are the values of x when $y = 6$?

9. A rectangular tank 32 cm by 24 cm contains water to a height of 15 cm. 72 marbles of diameter 3 cm each are dropped into the tank. Find the rise in the water level, giving your answer correct to the nearest millimetre.

10. Using a scale of 2 cm to represent 2 units on both axes, plot the graphs of $y = 2x + 2$ and $y + 3x = 10$. Use your graph to find the solution of the simultaneous equations $y = 2x + 2$ and $y + 3x = 10$.

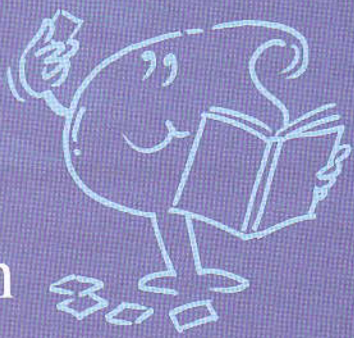
In this chapter, you will learn how to

- ***define: a set, an empty set, equal sets, finite subsets and proper subsets, universal sets, and complement of a set;***
- ***show the relationship between sets by using Venn diagrams;***
- ***define the intersection and union of sets using Venn diagrams and identify the set shaded in a Venn diagram;***
- ***solve problems on classification and cataloguing;***
- ***express problems in set notations and use Venn diagrams to obtain solutions.***

National Day is a time for Singaporeans to come together, both locally and abroad, to celebrate our nation's growth and to reflect on the nation's achievements and challenges for the future.

2006 will mark 41 years of independence … Singapore aims to remake itself into a … cosmopolitan city and an internation… that not only offers numerous … success but also … work and pl… call ho…

Set Language and Notation



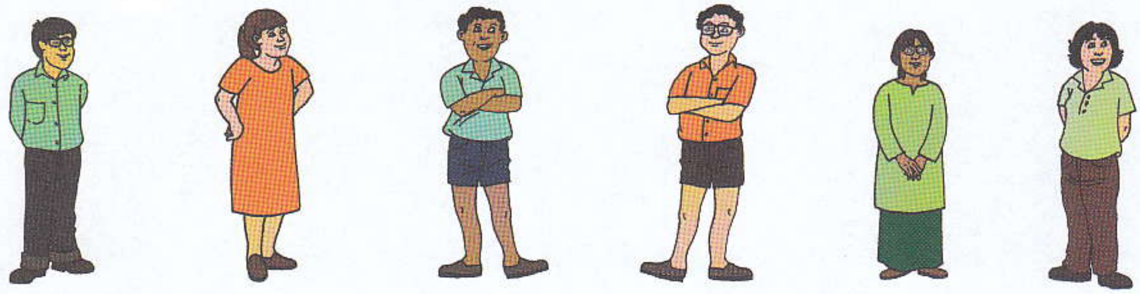
Introduction

In everyday life, as well as in mathematics, we are often more concerned not with individual objects but with a collection of objects. We use many different collective terms to describe different collections of objects. For example, we speak of a collection of paintings, a fleet of cars, an army of soldiers, a pair of gold-plated orchid earrings, a set of right-angled triangles, a crowd of people, etc. To simplify things, mathematicians use the term "set" to describe all these.

The photograph shows a set of Singapore commemorative stamps celebrating the nation's 41 years of independence in 2006.

Introduction to Sets

Look at the teenagers below.



Kenneth Min Hui Sanjay Zhao Yong Siti Delia

Fig. 10.1

How can we group the teenagers?

One easy way is to group them by gender. Kenneth, Sanjay and Zhao Yong are in the group of boys while Min Hui, Siti and Delia are in the group of girls.

A group or collection of objects is called a **set**. The individual objects in a set are called the members or **elements** of the set.

How do we define a set? Let's use the example above.

1. Listing the elements

The set of all the teenagers above can be defined by listing all its elements in the following way:

{Kenneth, Min Hui, Sanjay, Zhao Yong, Siti, Delia}

and the set of all the boys above is

{Kenneth, Sanjay, Zhao Yong}.

2. Describing the elements

A set can also be defined by describing the elements clearly, e.g.

$A = \{x: x$ is a teenager$\}$,
$B = \{x: x$ is a teenager who wears glasses$\}$.
or simply $B = \{$teenagers who wear glasses$\}$.

This notation is useful when there are many elements in the set.

In set language, the Greek letter epsilon, $\in$ means "is a member of" and $\notin$ means "is not a member of". For example, if $B = \{x: x$ is a teenager who wears glasses$\}$, then Kenneth $\in B$ and Min Hui $\notin B$. Usually, we use capital letters to denote a set and small letters to denote members of the set.

FOR YOUR Information

George Cantor (1845-1918), renowned German mathematician, is regarded as the founder of "The Set Theory" and the one who introduced this theory to the mathematical world. Today, the set theory remains as one of the foundations of many advanced mathematical works.

Do it Yourself

Define three other sets for the above example, using the methods above.

Explorations

You are to work in groups of 4.

Activity A

1. Write out a list of CCAs that your school is offering to the pupils. Write as many as you know. You are given 5 minutes to complete this task.
2. Exchange the list with a neighbouring group.
3. Classify the CCAs drawn up by your friends according to: (a) Sports and Games, (b) Clubs and Societies and (c) Uniform Groups.
4. Exchange the list that you have classified with the next group and check if the classification is properly done.
5. Write in set notation for each of the sets.

Activity B

1. Write out a list of festivals and religious ceremonies celebrated by the different communities in Singapore. Write as many as you know. You are given 10 minutes to complete this task.
2. Exchange the list with a neighbouring group.
3. Classify the festivals and religious ceremonies drawn up by your friends according to festivals and religious ceremonies observed by the (a) Chinese, (b) Malays, (c) Indians and (d) Eurasians.
4. Exchange the list that you have classified with the next group and check if the classification is properly done.
5. Explain the significance of each of the festivals and religious ceremonies to your classmates if they are not familiar with them.
6. Write in set notation for each of the sets.

Example 1

List all the elements in each of the following sets.
(a) The set of vowels in the English alphabet.
(b) $S = \{x: x$ is a positive odd number less than $10\}$.
(c) $T = \{$letters in the word "WOOD"$\}$.

Solution

(a) The set of English vowels = {a, e, i, o, u}.
(b) $S = \{1, 3, 5, 7, 9\}$.
(c) $T = \{W, O, D\}$

Notice that when listing the elements of a set, identical elements are not repeated.

Example 2

Describe the following sets in words.
(a) A = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}
(b) $B = \{1, 2, 3, 4, 5\}$

Solution

(a) A is the set of days in the week.
(b) B is the set of the first five natural numbers, or the set of natural numbers less than 6.

Example 3

State whether each of the following collections is a well-defined set. Give a reason for your answer.
(a) {A book well-liked by my classmates}
(b) {A naughty pupil in my class}
(c) {A Mathematics teacher in my class}

Solution

(a) No, because a book may be well-liked by some, but not others.
(b) No, because some teachers and pupils may consider a certain pupil naughty, while others may not.
The definition of a naughty pupil is unclear.
(c) Yes, because it is clear whether someone is a mathematics teacher in the class.

Exercise 10a

1. List the elements of the following sets.
   (a) {x: x is the month of the year beginning with the letter J}
   (b) {x: x is an odd number between 10 and 18}
   (c) {x: x is the first five consonants of the English alphabet}
   (d) {x: x is the day of the week beginning with the letter T}
   (e) {x: x is an even number on a clock face}
   (f) {x: x is the month of the year with fewer than 30 days}
   (g) {x: x is a Mathematics teacher in my school}
   (h) {x: x is a pupil in my class who is taller than 1.6 m}

2. Describe the following sets.
   (a) $A = \{2, 4, 6, 8, 10\}$
   (b) $B = \{2, 4, 6, 8, 10, \ldots\}$
   (c) $C = \{a, b, c, d, e\}$
   (d) D = {basketball, football, rugby, softball, squash}
   (e) E = {apples, oranges, bananas, pears, pineapples}

3. State whether each of the following is true (T) or false (F).
   (a) $5 \in \{1, 3, 5, 7\}$
   (b) $4 \in \{1, 3, 5, 7\}$
   (c) bus $\in$ {b, u, s}
   (d) b $\in$ {b, u, s}

4. Which of the following statements are true (T) and which are false (F)?
   (a) $5 \notin \{3, 7, 11, 14\}$
   (b) i $\in$ {a, i, j, k}
   (c) $4 \notin$ {x: x is an even number}
   (d) {S, C, O, H, L} $\in$ {x: x is a letter of the word "SCHOOL"}
   (e) $\left(3\frac{3}{4} + 1\frac{1}{4}\right) \in$ {x: x is an even number}
   (f) $\{3\} \in \{1, 2, 3, 4, 5\}$

5. State whether each of the following collections is a well-defined set. Give a reason for each answer.
   (a) {A pupil in my class who has two brothers}
   (b) {A pupil in my class who is shy}
   (c) {A TV actor who is well-liked by my classmates}
   (d) {A dish well-liked by my family members}
   (e) {A textbook used in my school}
   (f) {The prettiest TV actress in Singapore}

6. Pick out the odd element in each of the following sets and describe the remaining elements in the set.
   (a) {Malaysia, Singapore, Cambodia, the Philippines, Vietnam, Laos, Myanmar, Brunei, Thailand, Indonesia, China}
   (b) {rubber, coconut, apple, mango, rambutan, orange}
   (c) {4, 9, 1, 25, 16, 49, 20, 36}
   (d) {8, 1, 64, 75, 27}
   (e) {mode, mean, pie chart, median}

7. List the elements of the following sets.
   (a) {Vowels in the word "MATHEMATICS"}
   (b) {The seven colours of the rainbow}
   (c) {Multiples of 9 which are less than 50}
   (d) {Colours in your school flag}
   (e) {Public holidays in Singapore}
   (f) {Even numbers which are between 10 and 23}

Number of Elements in a Set

How many elements are there in the set of alphabet in the English language? We know very well that the answer is 26.

Usually, for a set E, we denote the number of elements in E as $n(E)$.

If E represents the set of alphabet in the English language, i.e.

E = {letters in the English alphabet}, $n(E) = 26$.

How many boys in your class have pigtails? How many girls in your class sport a moustache? Let P and M represent the sets respectively. Notice that these sets have no elements in them, i.e. $n(P) = 0$ and $n(M) = 0$.

We call a set with no elements the **null set** or **empty set**. It is represented by the symbol $\varnothing$ or denoted by { }.

Thus P = {boys in class with pigtails} = $\varnothing$ or { }

M = {girls in class with a moustache} = $\varnothing$ or { }.

Do it Yourself

Define each of the following sets and write down the number of elements in each set, using set notation.

(a) Channels of our T.V. Broadcasting station.
(b) English letters in your name.
(c) Months with 27 days.
(d) Dogs with six legs.
(e) Stalls in your school canteen.
(f) Gymnasts in your class.

Exercise 10b

1. Write in set notation the number of elements in each of the following sets.
   (a) Months in the year
   (b) Odd numbers between 10 and 26
   (c) Pupils in your class who are taller than you
   (d) Singaporeans who have been to the Moon
   (e) Colours in your school flag
   (f) Rainbow colours
   (g) Horoscope signs

2. Writing in set notation, state which of the following sets are empty sets.
   (a) Polar bears living in the Sahara Desert
   (b) Cars with 3 doors
   (c) Buses with 50 seats
   (d) Bald men with a crew cut
   (e) Boys who wear skirts to school
   (f) Singaporeans who have landed on the Moon
   (g) Girls weighing 200 kg each
   (h) Orators who cannot talk well
   (i) Odd numbers that are divisible by 4
   (j) Tigers in Singapore
   (k) Pandas in Singapore
   (l) Members of Parliament in Singapore who are below 21 years old.
   (m) High jumpers who have cleared 3 m
   (n) Singaporeans who had won a gold medal at the Olympics
   (o) Triangles having three equal sides
   (p) Parallelograms having five vertices
   (q) Quadrilaterals having three obtuse angles

Venn Diagrams

We have studied two ways of expressing a set:
(a) by listing its elements within braces,
(b) by stating its chief characteristics in words.

Besides these two methods, we can also express a set by means of a diagram called a **Venn diagram** (see Fig. 10.2).

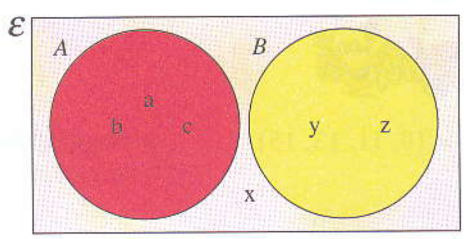


Fig. 10.2

We shall study the Venn diagram in greater detail.

From Fig. 10.2, we can see that the rectangle contains set A and set B.

The rectangle is the set that contains all the sets in discussion. It is called the universal set and is denoted by $\mathcal{E}$.

Complement of a Set

Suppose $\mathcal{E}$ = {integers from 1 to 10}, A = {odd numbers in the range 1 to 10} and B = {even numbers in the range 1 to 10},

$$\mathcal{E} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$
$$A = \{1, 3, 5, 7, 9\}$$
$$B = \{2, 4, 6, 8, 10\}$$

We find that any element in $\mathcal{E}$ is either an odd number or even number, i.e. either in set A or set B. These two sets have no elements in common, so they have no common area.

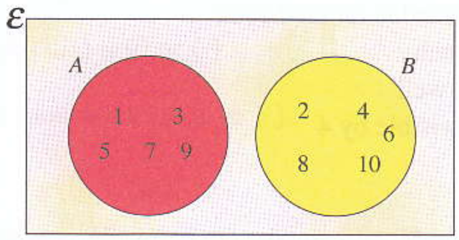


Fig. 10.3

Elements that are in $\mathcal{E}$ but not in A, are members of a set called the **complement** of A (denoted as A'). In the above example, $A' = B$.

Example 4

If $\mathcal{E} = \{3, 5, 7, 9, 11, 13, 15\}$, $A = \{3, 5, 7\}$, $B = \{9, 11, 13\}$ and $C = \{5, 7, 9, 11, 13, 15\}$, list the elements of the sets A', B' and C'.

Solution

$A' = \{9, 11, 13, 15\}$ $B' = \{3, 5, 7, 15\}$ $C' = \{3\}$

Example 5

If $\mathcal{E} = \{x: 0° < x < 180°\}$ and $A = \{x: 0° < x \leq 90°\}$, write A' in set notation.

Solution

$A' = \{x: 90° < x < 180°\}$ or
$A' = \{\text{obtuse angles}\}$.

Subsets

Consider $\mathcal{E}$ = {integers from 3 to 10}, A = {odd numbers in the range 3 to 10} and B = {prime numbers in the range 3 to 10}.

$\mathcal{E} = \{3, 4, 5, 6, 7, 8, 9, 10\}$
$A = \{3, 5, 7, 9\}$
$B = \{3, 5, 7\}$

Notice that each member of B is also a member of A. We say that B is a **subset** of A. The Venn diagram is drawn as follows:

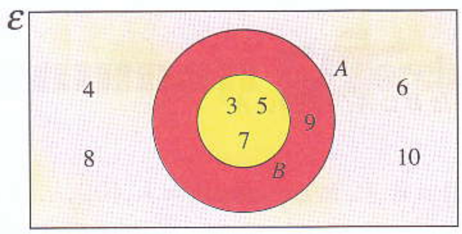


Fig. 10.4

We use the notation $B \subseteq A$ or $B \subset A$ to denote B is a subset of A. $\subseteq$ expresses the idea "includes" or "contains" and $\subset$ shows proper inclusion. Thus $B \subset A$ means that B is a proper subset of A implying that B has at least one element fewer than A.

Now consider the sets:

C = {prefects in your class}
D = {pupils in your class}
E = {pupils in your standard or grade level}
F = {pupils in your school}

Do you notice that all the members of C are also members of D, all the members of D are members of E, and all the members of E are also members of F?

We write $C \subset D \subset E \subset F$. Is $C \subset F$? Is $D \subset F$? Is $\varnothing \subset C$?

Mathematicians regard $\varnothing$ as a subset of every set.

Just for fun

Adam, Bob, Charles and David are each proficient in two out of four languages (English, German, French, Chinese) and only one of the four languages can be understood by three of them. They do not have a common language. Find what languages each of them is proficient in, taking into consideration the following situations.

(1) Bob does not understand English but he acts as an interpreter for Adam and Charles when they converse with each other.
(2) Adam speaks German, but not David. Yet they can converse with each other without any difficulty.
(3) Bob, Charles and David do not have a common language.
(4) None of them is able to use both German and French to communicate with one another.

Thinking Time

Does $A \subset B \Rightarrow A \subseteq B$?

Does $A \subseteq B \Rightarrow A \subset B$?

Notes

As an analogy, consider the following:

In the whole number system, $0 \in$ {all whole numbers}.
Similarly, in set notation, $\varnothing \in$ {any sets}.

Hence we can regard $\varnothing$ as a subset of every set.

Example 6

If $A = \{car, c, a, t\}$ and $B = \{c, a, r, t\}$, is $A \subseteq B$?

Solution

The word 'car' is an element of A but not of B, although the individual letters of the word belong to B.

Thus A is not a subset of B and we write $A \nsubseteq B$. We also use the symbol $A \not\subset B$ to denote that A is not a proper subset of B. Is $B \subseteq A$?

Example 7

If $A = \{a, b, c, d\}$ and $B = \{d, b, a, c\}$, is $A \subseteq B$? Is $B \subseteq A$? Is $\varnothing \subseteq A$?

Solution

Since every element of A is also an element of B, $A \subseteq B$. Similarly, every element of B is also an element of A, thus $B \subseteq A$.

Note that A and B contain exactly the same elements. What can you say about A and B?

Since $\varnothing$ is a subset of every set, $\varnothing \subseteq A$.

Example 8

List the subsets of the set {a, b}.

Solution

The subsets are $\varnothing$, {a}, {b} and {a, b}.
Can you list the subsets of {a, b, c}?

Example 9

If I = {real numbers}, P = {prime numbers} and C = {composite numbers}, draw a Venn diagram with I as the universal set, $\mathcal{E}$. Describe the relationship between the sets using set notation.

Solution

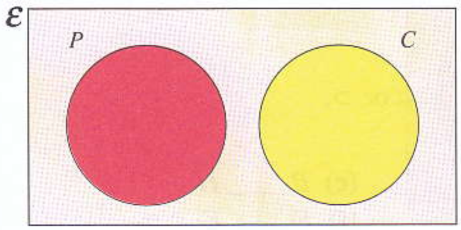


Fig. 10.5

$P \subset I$, $C \subset I$, $P = C'$

Exercise 10c

1. If $\mathcal{E} = \{1, 2, 3, \ldots, 20\}$, A = {multiples of 2}, B = {multiples of 4} and C = {multiples of 3}, list the elements of the following.

(a) A **(b)** B **(c)** C

(d) A' **(e)** B' **(f)** C'

2. If $\mathcal{E} = \{30, 31, 32, \ldots, 45\}$, list the complements of the following sets.

$A = \{33, 37, 39, 42, 44\}$

$B = \{30, 31, 32, 33, 34, 36, 37, 38, 39, 40, 41, 42, 45\}$

C = {non-prime numbers}

D = {multiples of 4}

E = {multiples of 3}

3. State whether each of the following statements below is true (T) or false (F).

(a) If $\mathcal{E} = \{a, b, c, d, e\}$ and $A = \{a, b, c\}$, then $A' = \{d, e\}$.

(b) If $a \in A$ and $A \subseteq B$, then $a \in B$.

(c) If $A \subset B$ and $B \subseteq C$, then $A \subseteq C$.

(d) If $a \in A$ and $a \in B$, then $A = B$.

(e) If $\varnothing = A'$, then $A = \mathcal{E}$.

4. If the universal set, $\mathcal{E} = \{1, 2, 3, \ldots, 10\}$ and $A = \{\text{prime numbers}\}$, $B = \{\text{even numbers}\}$ and $C = \{\text{multiples of 5}\}$, list the elements of A, B, C, A', B' and C'.

5. Given the following sets:
$A = \{1, 2, 3, 4\}$,
$B = \{a, b, c, d\}$,
$C = \{a, o, u, z\}$,
$D = \{a, e, i, o, u\}$,
$E = \{2, 3, 4\}$,
$F = \{4, 3, 2\}$,
$G = \{\text{the vowels in the English alphabet}\}$,
$H = \{\text{the first four letters in the English alphabet}\}$,
$I = \{\text{the first four counting numbers}\}$,

fill in the blanks with one of the following symbols: $=$, $\neq$, $\subseteq$, $\supseteq$, $\subset$ or $\supset$.
There may be more than one answer in certain cases.

(a) A ____ B **(b)** A ____ F **(c)** B ____ H
(d) H ____ C **(e)** G ____ I **(f)** F ____ E
(g) F ____ I **(h)** D ____ G **(i)** C ____ I
(j) I ____ A **(k)** E ____ A **(l)** G ____ A

6. List all the subsets of the following sets.
(a) $\{1, 2\}$ **(b)** {pen, ink, ruler}
(c) {Singapore, Malaysia} **(d)** $\{a, e, i, o\}$

7. If $A = \{a, \{a\}, b, \{c\}, d\}$, state whether each of the following is true (T) or false (F).
(a) $a \in A$ **(b)** $\{a\} \in A$
(c) $c \in A$ **(d)** $\{d\} \in A$
(e) $\{a, \{a\}\} \in A$ **(f)** $\{a\} \subseteq A$
(g) $\{\{a\}\} \subseteq A$ **(h)** $\{a, b\} \in A$
(i) $\{a, b\} \subseteq A$ **(j)** $\{b, \{c\}, d\} \subseteq A$
(k) $\{a, \{a, c\}, d\} \subseteq A$

8. If $A = \{10, 20, 40, 60, 80\}$, list the elements of the following subsets of A.
(a) {numbers divisible by 4} **(b)** {numbers divisible by 6}
(c) {numbers divisible by 8} **(d)** {numbers divisible by 7}

9. List three subsets for each of the following sets.
(a) A = {Japanese cars}
(b) B = {animals in Singapore's Mandai Zoo}
(c) C = {trees found in the Botanic Gardens}
(d) D = {pupils in your class}

10. If $\mathcal{E} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2\}$ and $B = \{1, 2, 3, 4\}$, draw a Venn diagram to represent the above sets and to illustrate their relationships.

Intersection of Sets (∩)

The **intersection** of sets A and B is the set of elements which are common to both A and B. It is denoted by $A \cap B$.

If $\mathcal{E} = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 2, 3, 5\}$ and $B = \{2, 3, 4, 6\}$, then $A \cap B = \{2, 3\}$, since 2 and 3 belong to both A and B. The relationship is illustrated by the Venn diagram in Fig. 10.6.

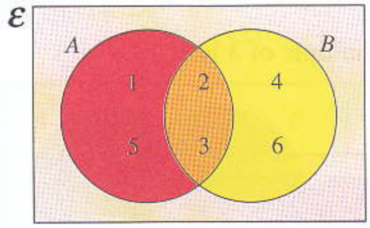


Fig. 10.6

Note that the overlapping region represents the set $A \cap B$.

Example 10

Draw a labelled Venn diagram, place the elements in the appropriate regions and identify $A \cap B$ in each of the following cases. The universal set $\mathcal{E}$ is $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

(a) $A = \{1, 2, 3, 5, 7\}$, $B = \{4, 6\}$.
(b) $A = \{\text{multiples of } 2\}$, $B = \{\text{multiples of } 3\}$.
(c) $A = \{\text{multiples of } 3\}$, $B = \{\text{prime numbers}\}$.
(d) $A = \{\text{multiples of } 3\}$, $B = \{\text{multiples of } 6\}$.

Solution

(a) The sets A and B have no common elements.

$\therefore \; A \cap B = \varnothing$

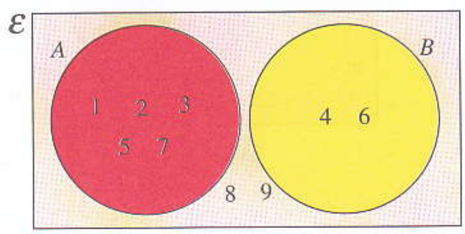


Fig. 10.7

(b) The only number which is a multiple of 2 and 3 is 6.

$\therefore \ A \cap B = \{6\}$

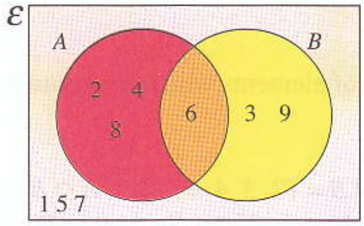


Fig. 10.8

(c) The only prime number which is a multiple of 3 is 3.

$\therefore A \cap B = \{3\}$

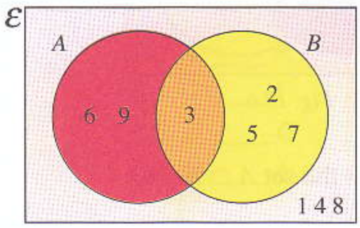


Fig. 10.9

(d) All multiples of 6 are also multiples of 3.

$\therefore \ B \subseteq A$ and $A \cap B = \{6\}$

i.e. $A \cap B = B$

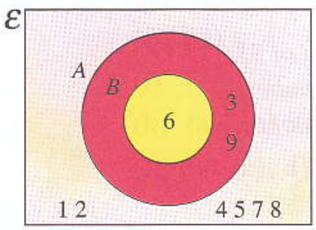


Fig. 10.10

Union of Sets (∪)

The **union** of sets A and B is the set of elements which are in A, or in B, or in both A and B. It is denoted by $A \cup B$.

If $\mathcal{E} = \{1, 2, 3, \ldots, 10\}$, $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6, 7\}$, then $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$.

Note that 3 and 4 are written only once in $A \cup B$. The union of the two sets A and B is represented by the green region in the Venn diagram shown.

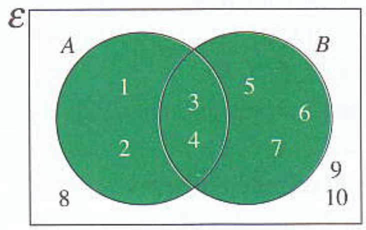


Fig. 10.11

Example 11

If $\mathcal{E}$ = {letters from h to p in the alphabet}, A = {letters of the word "hip"} and B = {letters of the word "hop"}, draw a Venn diagram to represent these sets and find the following.

(a) $A \cup B$
(b) $A \cap B$

Solution

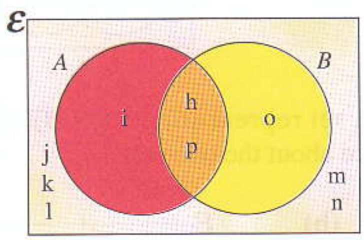


Fig. 10.12

(a) $A \cup B = \{h, i, p, o\}$
(b) $A \cap B = \{h, p\}$

Just for fun

1. *Jack has 9 coins that look similar. One of them is lighter than the others. Explain how he can select the lighter coin with only 2 weighings, using a balance.*

2. *Jane has two coins. Their total value is 25 cents. If one of them is not a five-cent coin, what is the value of the other?*

Exercise 10d

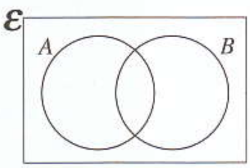
1. Draw labelled Venn diagrams to illustrate the following sets, placing the elements in the appropriate regions. In each case, identify $A \cap B$. $\mathcal{E}$ = {letters of the English alphabet}.
   (a) A = {a, b, c, e, f}, B = {b, c, o, p, q}.
   (b) A = {a, c, e, g}, B = {p, q, r}.
   (c) A = {x, y, z, m, n}, B = {consonants in the word"money"},
   (d) A = {consonants in the word "mathematics"}, B = {consonants in the word "statistics"}.
   (e) A = {letters of the word "universal"}, B = {letters of the word "probability"}.
   (f) A = {vowels in the word "transformations"}, B = {vowels in the word "combinations"}.

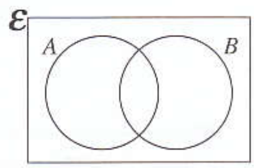
2. Draw labelled Venn diagrams to illustrate the following sets, placing the elements in the appropriate regions, and in each case, find $A \cup B$. $\mathcal{E} = \{1, 2, 3, ..., 9\}$
   (a) $A = \{1, 2, 3, 4\}$, $B = \{3, 5, 7, 9\}$.
   (b) $A = \{1, 3, 5, 7, 9\}$, $B = \{2, 4, 6, 8\}$.
   (c) $A = \{4, 8\}$, $B = \{2, 4, 6, 8\}$.
   (d) A = {multiples of 3}, B = {prime numbers}.
   (e) A = {multiples of 4}, B = {multiples of 2}.

3. If A = {durian, mango, pineapple} and B = {durian, rambutan, mango, soursop}, find $A \cup B$ and $A \cap B$.

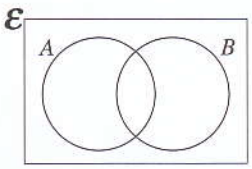
4. Find the union and intersection of each of the following pairs of sets.
   (a) $A = \{3, 6, 9, 12\}$, $B = \{6, 8, 9\}$.
   (b) C = {a, b, x, y}, D = {m, n, o, p}.
   (c) E = {monkey, goat, lion}, F = {tiger, goat}.
   (d) G = {a, m, k, y}, $H = \varnothing$.

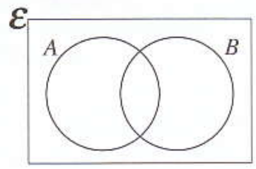
*5. Copy and shade, on diagram (a), the set representing $(A \cup B)'$ and on (b), the set representing $A' \cap B'$. What do you notice about these two sets?

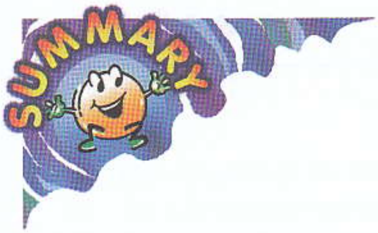
(a) 

(b) 

*6. Copy and shade, on diagram (a), the set representing $(A \cap B)'$ and on diagram (b), the set representing $A' \cup B'$. What do you notice about the two sets?

(a) 

(b) 



1. A set is a collection of objects which are clearly defined. The objects belonging to a set are called its **elements**.

2. A set can be defined by
   (a) listing its elements within braces, e.g, {Ahmad, Ali, John},
   (b) stating its characteristics in words, e.g.{letters in the alphabet},
   (c) drawing a Venn diagram.

3. The **empty** or **null** set is the set containing no element. It is denoted by $\varnothing$.

4. Two sets are **equal** if they have exactly the same elements.

5. A is said to be a **subset** of B, written as $A \subseteq B$, if all the elements of A are also the elements of B.

6. $\varnothing$ is the subset of every set.

7. The **complement** of the set A, written as A', is the set of elements in the universal set, usually denoted by $\mathcal{E}$, which are not members of A.

8. The **intersection** of set A and set B, written as $A \cap B$, is the set of elements common to both A and B.

9. The **union** of set A and set B, written as $A \cup B$, is the set of elements which are in A, or B or in both A and B.

Review Examples 10

Example 1

If $\mathcal{E} = \{x: x \text{ is an odd number}, 10 \leq x \leq 40\}$, $A = \{x: x \text{ is a prime number}\}$ *and* $B = \{y: y \text{ is a multiple of } 13\}$, *list the members of the following.*

(i) $A \cap B$

(ii) $A \cup B$

(iii) $A \cap B'$

Solution

$A = \{11, 13, 17, 19, 23, 29, 31, 37\}$, $B = \{13, 39\}$

(i) $A \cap B = \{13\}$

(ii) $A \cup B = \{11, 13, 17, 19, 23, 29, 31, 37, 39\}$

(iii) $B' = \{11, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37\}$

$\therefore\ A \cap B' = \{11, 17, 19, 23, 29, 31, 37\}$

Example 2

$A = \{(x,y): (x,y)$ *lies on the line* $y = 2x + 3\}$ *and* $B = \{(x,y): (x,y)$ *lies on the line* $y = kx + h\}$.

If $n(A \cap B) = 0$, *state the value of k and give a possible value for h.*

Solution

A is the set of points on the straight line $y = 2x + 3$ with gradient 2 and y-intercept 3. B is the set of points on the line $y = kx + h$. Since $n(A \cap B) = 0$, the two lines do not intersect. Therefore, they must be parallel, i.e. $k = 2$. h can take any value except 3.

Do you know why?

Review Questions 10

1. List the elements of the following sets.
   (i) $\{x: x$ is a natural number, $x = 3y$ and $2 \leq y \leq 10\}$
   (ii) $\{x: x$ is a whole number and $3x + 2 = 14\}$
   (iii) $\{x: x$ is a positive integer and $2x + 1 \leq 12\}$
   (iv) $\{x: x$ is a rational number and $2x - 3 = 8\}$

2. State which of the following sets are empty sets.
   (i) Long jumpers who have jumped more than 8 metres.
   (ii) Pentagons having 4 acute angles.
   (iii) Polygons having 5 obtuse angles.
   (iv) Runners who had run 100 m in 9.6 seconds.
   (v) Points of intersection of the straight lines $3x + y = 5$ and $2y = 10 - 6x$.

3. State which of the following pairs of sets are equal.

(i)	$A = \varnothing$,	$B = \{0\}$.
(ii)	$A = \{1, 3, 5\} \cap \{2, 4\}$,	$B = \{x: x$ is prime and $5x - 4 = 1\}$.
(iii)	$A = \{x: x^2 = 9\}$,	$B = \{x: 2x - 5 = 1\}$.
(iv)	$A = \{\text{rhombus}\} \cap \{\text{rectangles}\}$,	$B = \{\text{squares}\} \cap \{\text{parallelograms}\}$.
(v)	$A = \{1, 2, 3, 4\}$,	$B = \{4, 3, 2, 1, 4\}$.

4. If $A = \{a, e, i, o, u\}$ and $B = \{l, m, n, o, p, q\}$, state which of the following statements are true.

(i) $a \in B$	(ii) $i \in B$	(iii) $o \notin A$
(iv) $1 \notin A$	(v) $n \notin A$	(vi) $e \in A$
(vii) $u \in A$	(viii) $m \in B$	(ix) $i \notin A$

5. Which of the following statements are true (T) and which are false (F)?
   (i) $3 \in \{3, 5, 7\}$
   (ii) $7 \notin \{2, 3, 6\}$
   (iii) $c \notin \{a, e, i, o, u\}$
   (iv) $6 \notin \{x: x$ is an even number$\}$
   (v) $15 \notin \{x: x$ is a prime number$\}$
   (vi) If $A = \{x: 2 < x \leq 9, x$ is an integer$\}$, then $2 \in A$.
   (vii) If $A = \{1, 2, 3, 4\}$ and $B = \{$whole numbers less than 4$\}$, then $A = B$.
   (viii) If $A = \{x: x$ is a positive even integer and prime$\}$, then $A = \varnothing$.
   (ix) If A is an empty set, then $A = \{\varnothing\}$.
   (x) The empty set is the only set without a subset.
   (xi) If $A \subseteq B$, then $A' \subseteq B'$.
   (xii) If $A \subseteq B$ and $B \cap C = \varnothing$, then $A \cap C = \varnothing$.

6. State whether each of the following statements is true (T) or false (F).

(i) If $A \cap B = \varnothing$, then $(A \cup B) \cap (A \cap B) = \varnothing$.

(ii) If $\mathcal{E}$ = {rational numbers}, $A = \{x: x \geq 4\}$, $B = \{x: x \leq 7\}$ and $C = \{4, 5, 6\}$, then $C \subseteq (A \cap B)$.

(iii) If $A \cup B = B$, then $B \subseteq A$.

(iv) If A is a subset of B, then $A \cup B = B$.

(v) If $A = \{0, 1, 2, 0, 2, 2, 1, 1\}$, $B = \{2, 1, 0\}$, then $A = B$.

(vi) If A = {rectangles with five vertices}, B = {odd numbers that are divisible by 12}, then $A = B$.

(vii) There are 32 subsets in the set {a, e, i, o, u}.

(viii) If $A \cap B = A$, then $A \cup B = B$.

(ix) If A, P and Q are sets, then $A \cap P = A \cap Q \Rightarrow P = Q$.

7. **(i)** If $n(P) = 46$, $n(Q) = 24$ and $P \cap Q = \varnothing$, find $n(P \cup Q)$.

(ii) If $n(R) = 43$, $n(S) = 74$ and $R \subseteq S$, find $n(R \cap S)$.

8. If $\mathcal{E} = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 4\}$ and $B = \{1, 4, 5\}$, list the elements of the set $(A \cup B) \cap (A \cap B)'$.

9. If $\mathcal{E}$ = {integers from 1 to 12, both inclusive}, P = {prime numbers} and Q = {odd numbers}, list the elements of the following.

(i) $P \cap Q$ **(ii)** $P \cup Q$ **(iii)** $P' \cup Q'$

(iv) $P' \cap Q'$ **(v)** $P \cap Q'$

10. The universal set is the set of the integers from 1 to 25, both inclusive. The sets P, Q and R are defined as

$P = \{ p: p \text{ is a multiple of } 2 \}$
$Q = \{ q: q \text{ is a multiple of } 3 \}$
$R = \{ r: r \text{ is a multiple of } 9 \}$

Illustrate the relationship of P, Q and R in a Venn diagram and mark the numbers in each region.

11. If $A \subseteq B$, $n(A \cap C) = 0$ and $n(B \cap C) > 0$, illustrate with a Venn diagram the possible relationship(s) between A, B and C.

12. The sets A, B and C satisfy the following three conditions:

$$B \subseteq A,\ A \cap C \neq \varnothing,\ B \cap C = \varnothing.$$

Represent these sets on a clearly-labelled Venn diagram.

13. Given that $\mathcal{E} = \{x: x \text{ is an integer and } 20 \leq x \leq 100\}$, $A = \{x: x \text{ divided by 12 leaves a remainder of 4}\}$, and $B = \{x: x \text{ divided by 14 leaves a remainder of 6}\}$,

(i) list the members of A and B,

(ii) find $n(A \cap B)$.

14. If A = {parallelograms}, B = {rectangles} and C = {squares}, simplify the following.

(i) $A \cup B$

(ii) $B \cap C$

(iii) $A \cap C$

15. If $\mathcal{E}$ = {rectangles of length x metres and breadth y metres where x and y are integers and $x > y$}, A = {rectangles each with an area of 24 m^2} and B = {rectangles each with length which is a multiple of 3 and an area of 24 m^2}, find the following.

(a) $n(A)$

(b) $n(B)$

(c) $n(A \cap B)$

16. Draw a Venn diagram to show the relationships between the following sets of plane figures.
$\mathcal{E}$ = {polygons}, A = {quadrilaterals}, B = {parallelograms}, C = {squares} and D = {triangles}.

17. The universal set $\mathcal{E}$ is the set of all triangles. Given that A = {isosceles triangles}, B = {equilateral triangles} and C = {obtuse-angled triangles}, illustrate the sets A, B and C within a Venn diagram.

18. If $A = \{2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5)$, find a set C such that $A \cup C = B$. Is C unique?

19. If A = {a, b, c, d} and B = {a, c}, find a set C such that $A \cap C = B$. Is C unique?

In this chapter, you will learn how to

- *interpret and analyse dot diagrams;*
- *interpret and analyse stem-and-leaf diagrams;*
- *find the mode, mean and median of a set of data;*
- *make appropriate use of mean, mode and median;*
- *calculate the mean for grouped data.*

CHAPTER

11

Statistics

Introduction

Do you know that the average Singaporean eats 65 kg of rice per year, that the average Singapore man wears size 7 shoes and that the average Singaporean family has two children? However, the method used to arrive at each of the above figures may differ.

Collection, Organisation and Interpretation of Data

Different people collect data for various purposes. For example, a man who wants to sell ice-cream might be interested in collecting data to determine the most popular flavour of ice-cream. The results of the GCE 'O' Level examinations are collected by the Ministry of Education to rank the schools so that parents and students are able to make more informed choices. A teacher might collect the marks scored by his/her students in a class test to analyse the academic performance of his/her class. In summary, people normally collect raw data and process it into information for reference or to make better decisions.

FOR YOUR INFORMATION

Do you know the word 'data' is plural? The singular word for data is 'datum'.

Consider the following scenario:

Mr Tan wants to know the general performance of his class of 30 students in a recent Mathematics test.

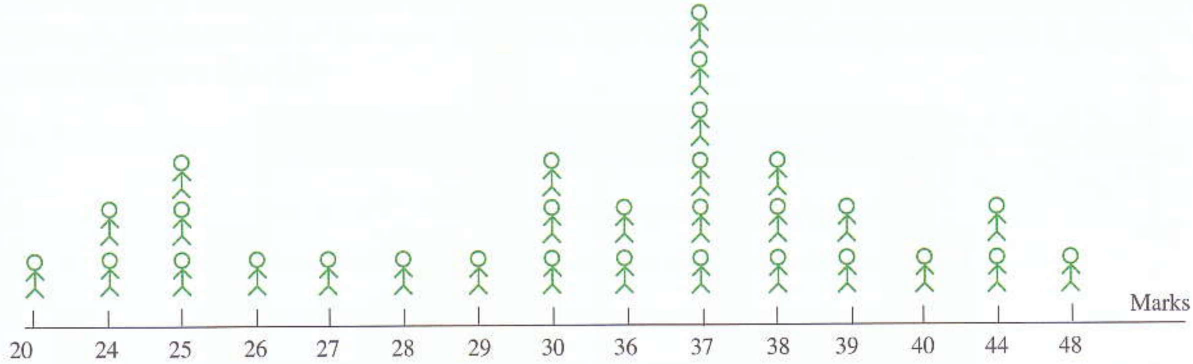
(1) To collect the data, he records the marks scored by each of his student using the class name list. The total marks for the class test is 50. The following shows the marks scored by 30 students in the class test:

36	37	38	25	37	39	36	30	24	38
37	28	48	25	44	25	38	37	44	29
37	37	30	40	39	24	26	27	20	48

(2) Since the marks are recorded using a name list, the marks are not in order. It is very hard for him to tell the performance of the class from the distribution of marks. Therefore, Mr Tan first organises the data by arranging the marks in ascending order (from the lowest to the highest):

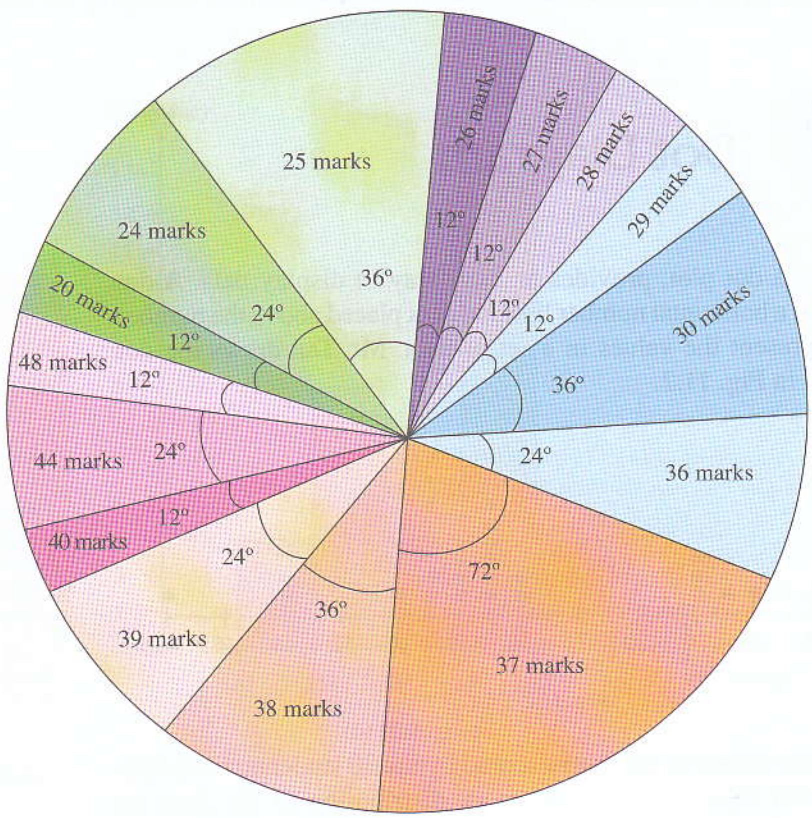
20	24	24	25	25	25	26	27	28	29
30	30	30	36	36	37	37	37	37	37
37	38	38	38	39	39	40	44	44	48

(3) Now the marks are in order. But it is still not easy to know the performance of the class, for example, the marks that most students score. So Mr. Tan decides to display the data graphically using a pictogram, a pie chart or a histogram which we have learnt in Book 1 (see Fig. 11.1a). Which graphical representation is not suitable to display this set of data?



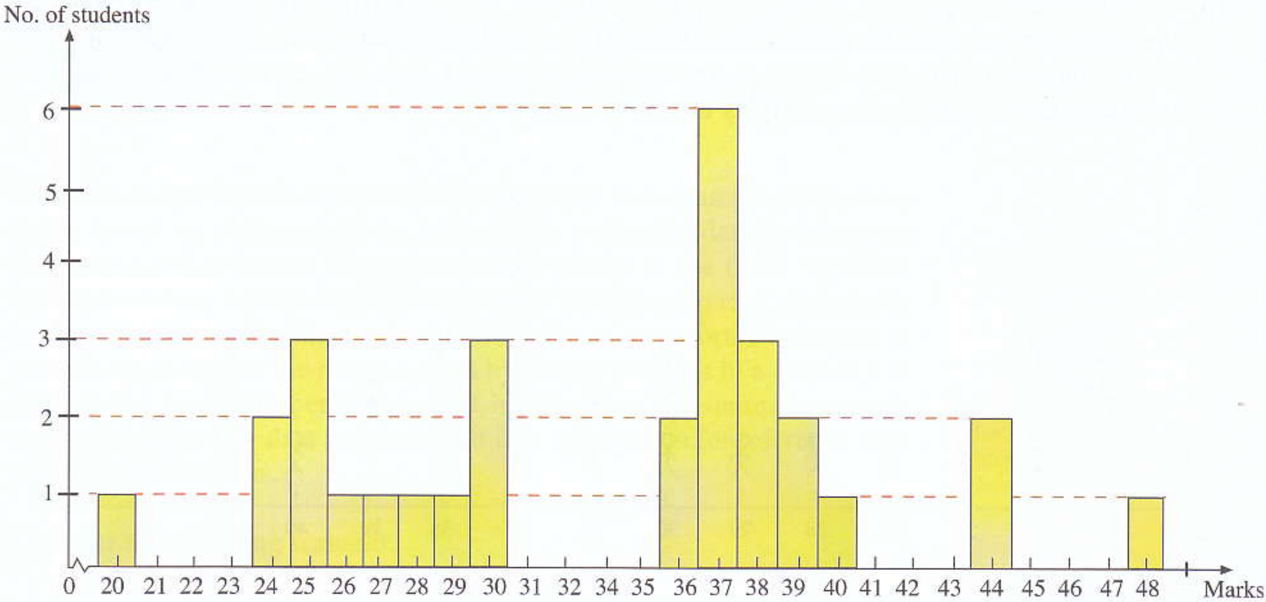
(a) Pictogram (Each ⚲ represents 1 student)

Fig. 11.1a



(b) Pie chart.

Fig. 11.1b



(c) Histogram.

Fig. 11.1c

Are there other types of graphical representation that we can use to display the data clearly?

Dot Diagram

A dot diagram, or a dot plot, provides an easy way to display data. A dot diagram consists of a horizontal number line and dots placed above the number line. The dots represent the values in a set of data. Mr. Tan draws the dot diagram as shown in Fig. 11.2.

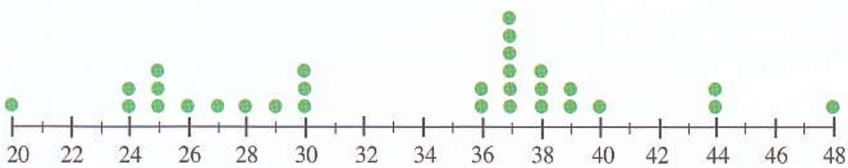
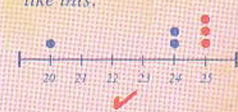


Fig. 11.2

(4) The dot diagram shows all the marks scored by the 30 students. From this dot diagram, Mr Tan **interprets** the mark distribution of his class as follows: the lowest score is 20; the highest score is 48; the most common score is 37; most students score between 24 and 30 marks (inclusive), and between 36 and 40 marks (inclusive).

The values on the number line should always be at ***equal intervals****. For example, given the data set 20, 24, 24, 25, 25, 25 the dot diagram should be like this:*



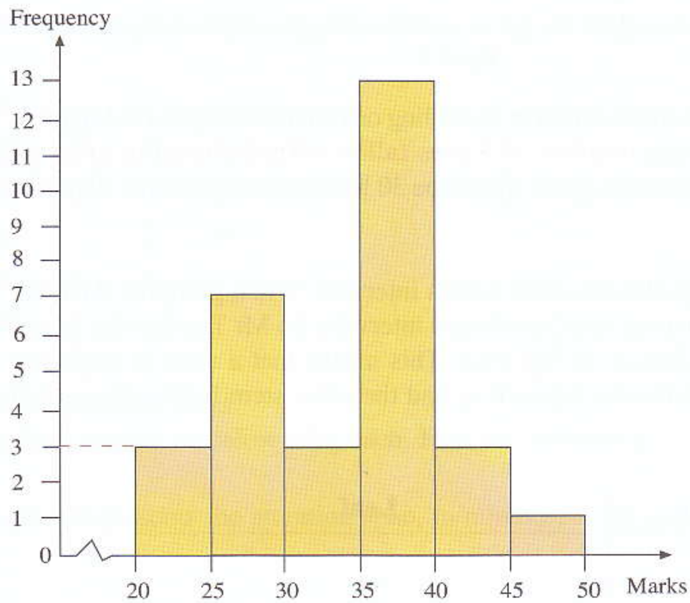
And NOT like this:

Grouped Data

Since the data is well-spread, recall that we have learnt in Book One how to organise the data by grouping them into class intervals of the same width (see Fig.11.3a) and then using a histogram to display the grouped data (see Fig.11.3b).

Marks	Frequency
$20 \leqslant x < 25$	3
$25 \leqslant x < 30$	7
$30 \leqslant x < 35$	3
$35 \leqslant x < 40$	13
$40 \leqslant x < 45$	3
$45 \leqslant x < 50$	1
	Total Frequency = 30

(a) Grouped Frequency Table



(b) Histogram

Fig. 11.3

From the grouped frequency table and the histogram, we can see very clearly that most students score between 35 and 39 marks (inclusive). But we cannot read the lowest and the highest scores: the individual data is 'lost'. To solve this problem, we use the stem and leaf diagram.

Stem and Leaf Diagram

The stem and leaf diagram, or stem plot, consists of a few stems, each with various number of leaves. Fig.11.4 shows Mr Tan's data displayed using a stem and leaf diagram.

Do not separate the leaves using commas.

Stem	Leaf
2	0 4 4 5 5 5 6 7 8 9
3	0 0 0 6 6 7 7 7 7 7 7 8 8 8 9 9
4	0 4 4 8

Fig. 11.4

In this case, the stem represents 'tens' and the leaf represents 'units'. For example, all the numbers 20, 24, 24, 25, 25, 26, 27, 28, 29 have a common stem 2. So the first number 20 is represented by the example in Fig. 11.5a and the second number 24 is represented by the example in Fig. 11.5b.

Stem	Leaf
2	0

(a)

Stem	Leaf
2	4

(b)

Fig 11.5

Note that the leaves are recorded in ascending order (from smallest to biggest). Always check that the **number of leaves tallies with the total number of data** collected. In this case, there should be 30 leaves since there are 30 marks recorded.

But Fig. 11.4 groups the data into 3 class intervals. When grouping data, we should try to group them into 5 to 8 class intervals. So Mr Tan decides to use the same grouped data as in Fig. 11.4. This means that a stem is displayed twice, i.e. one stem for the leaves 0-4, and the other stem for the leaves 5-9 (see Fig. 11.6).

Stem	Leaf
2	0 4 4
2	5 5 5 6 7 8 9
3	0 0 0
3	6 6 7 7 7 7 7 7 8 8 8 9 9
4	0 4 4
4	8

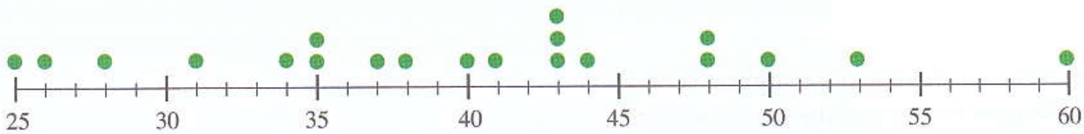
Fig. 11.6

Compare this stem and leaf diagram with the grouped frequency table and histogram in Fig.11.3. Not only can we see very clearly that most students score between 36 and 39 marks (inclusive), we can also see that the lowest score is 20 and the highest score is 48.

To summarise, the stem and leaf diagram can be used to display grouped data clearly, without any loss of individual data.

Exercise 11a

1. The following dot diagram represents the travel times, in minutes, from home to office of 20 company executives:



(a) What is the most common travel time?

(b) What is the percentage of executives who take less than half an hour to reach the office? Comment briefly on the data.

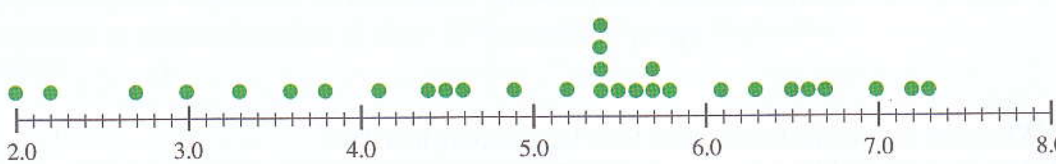
2. The following stem and leaf diagram represents the masses, in kg, of 40 boxes.

8	0	3	4												
8	6	6	6	7	8	8	9	9							
9	0	0	0	0	0	0	0	1	1	2	2	3	3	3	4
9	5	6	6	7	7	9									
10	0	1	2	4	4										
10	6	8	8												

(a) Write down the most common mass.

(b) 50% of the boxes have a mass of below a kg each. Find the value of a.

3. The following dot diagram represents the attention span, in minutes, of 30 pre-school children.



(a) What is the most common attention span?

(b) What is the percentage of children with attention spans falling below 6 minutes? Comment briefly on the data.

4. The following back-to-back stem and leaf diagram represents the scores of the students in two different schools for a common examination. In each school, 29 students took the examination.

ABC School Leaf	Stem	XYZ School Leaf
4 0	5	2 6 8 9
9 9 6	6	2 5 8 8 9 9
9 8 5 3 2 0	7	4 6 7 8 8 9
9 9 9 7 7 6 6 4 2	8	0 3 4 4 6 7 7
9 8 8 7 6 6 3 2 0	9	0 2 7 8
	10	0 0

(a) Which school had the highest scorer?
(b) Which school had the lowest scorer?
(c) Which school did better in the examination? Why?

5. The following is a record of the times, in minutes, John and Kevin used their mobile telephone per day for the past 20 days.

John Leaf	Stem	Kevin Leaf
2	1	0 5
6	2	
7	3	0 0 1 2 2 2 5 7
7	4	1 2 6 6 9 9
9 8	5	
5 5	6	
7 0	7	
9 9 8 3 3 4 1 1	8	2 2 4
8 1	9	9

(a) Who used his mobile telephone for the longest time in a day?
(b) Who used his mobile telephone for the shortest time in a day?
(c) Comment on the distribution of data for each boy.
(d) How would you judge which boy used his mobile telephone more frequently than the other?

Mode

We have just learnt how to organise statistical data and display them in a dot or stem-and-leaf diagram so that certain important information can be extracted from them.

For example, Mr Tan has found from Fig. 11.2 that the most common score in his class of 30 students is 37. In statistics, the **number which occurs most frequently** in a set of numbers is called the **mode** of the set of numbers. The mode is important to manufacturers who would like to produce most of their goods (e.g. shoes, shorts, skirts, etc.) in the most popular sizes so as to gain a higher percentage of the market share. The drink seller in the school canteen will be interested in the brand of drinks which is the most popular so that he/she can place more orders for this brand.

FOR YOUR INFORMATION

In July to September 2005, Channel 8 has the highest TV viewership of 50.6% as compared to other local free-to-air channels. Can you think of some reasons for the high viewership?

Example 1

(a) The lengths of 10 terrapins in mm are 63, 63, 75, 67, 69, 52, 50, 63, 56, 52. Find the mode.

(b) Suppose 2 more terrapins of lengths 70 mm and 52 mm respectively are added into the 10 terrapins. Find the mode.

Solution

(a) The mode of the lengths is 63 mm since it appears most often (3 times) among the ten lengths.

(b) The mode of the lengths are 52 mm and 63 mm since they occur the most frequently at an equal number of times (3 times each) among the twelve lengths.

We observe that a set of data may have more than one mode.

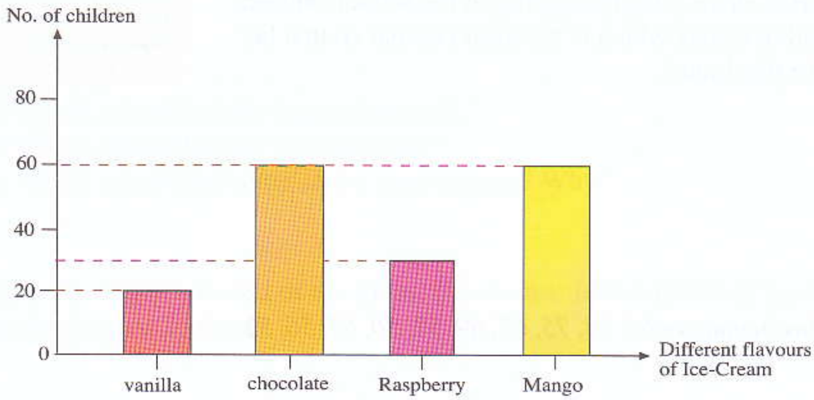
Example 2

Find the mode in each of the graphical represention below.

(a)

Stem	Leaf
2	0 4 4 5 5 5 6 7 8 9
3	0 0 0 6 6 7 7 7 7 7 7 8 8 8 4
4	0 4 4 8

(b)



(c)

Salary ($)	1000	1200	1500	2000
No. of Workers	2	5	3	1

Solution

(a) The mode is 37 since it appears 6 times, which is the highest frequency.

(b) The modes are chocolate and mango flavours because they both have the highest frequency of 60 chidren.

(c) The mode is $1200 because there are 5 workers (the highest frequency) who are given this salary.

The Mean

Let us consider the case of the scores of 30 students in Mr Tan's class again. Suppose Mr Tan would like to know about the "average score" of his class. He would then have to sum up all the marks of his students and then divide by the number of students in his class in order to obtain the "average score" per student. In statistics, this value is called the **mean**:

Notes

The mean is the most widely used average. But remember that the mode is also a type of average.

$$\textbf{Mean} = \frac{\textbf{Sum of data}}{\textbf{Number of data}}$$

Example 3

The salaries of 5 workers are given as follows:

\$2500, \$1200, \$1000, \$2500, \$1100
Find the mean salary.

Solution

$$\text{Mean salary} = \frac{\text{Total salary}}{\text{Total number of workers}}$$

$$= \frac{\$1000 + \$1100 + \$1200 + \$2500 + \$2500}{5}$$

$$= \frac{\$8300}{5}$$

$$= \$1660$$

Example 4

The stem and leaf diagram for marks in Mr Tan's class is as follows:

Stem	*Leaf*
2	*0 4 4 5 5 5 6 7 8 9*
3	*0 0 0 6 6 7 7 7 7 7 7 8 8 8 9 9*
4	*0 4 4 8*

Find the mean score.

Solution

$$\text{Mean Score} = \frac{\text{Sum of data}}{\text{Number of data}}$$

$$= \frac{20 + 24 \times 2 + 25 \times 3 + 26 + 27 + 28 + 29 + 30 \times 3 + 36 \times 2 + 37 \times 6 + 38 \times 3 + 39 \times 2 + 40 + 44 \times 2 + 48}{30}$$

$$= \frac{1005}{30}$$

$= 33.5$ marks

Example 5

The table below shows the salaries of 50 workers. Find the mean salary.

Salary ($)	*1000*	*1100*	*1200*	*2100*	*2500*
No. of Workers	*20*	*8*	*10*	*7*	*5*

Solution

$$\text{Mean Salary} = \frac{\text{Total salary}}{\text{Total number of workers}}$$

$$= \frac{\$1000 \times 20 + \$1100 \times 8 + \$1200 \times 10 + \$2100 \times 7 + \$2500 \times 5}{20 + 8 + 10 + 7 + 5}$$

$$= \frac{\$68\,000}{50}$$

$= \$1360$

For Your Information

The mean number of years of schooling for residents aged 25 years and above in Singapore in the year 2004 is 8.8 years.

In general, a set of data $x_1, x_2, x_3, \ldots, x_n$ occurring with the corresponding frequencies $f_1, f_2, f_3, \ldots, f_n$, is usually displayed in the form of a frequency table as follows:

x	x_1	x_2	x_3	$\ldots$	x_n
f	f_1	f_2	f_3	$\ldots$	f_n

The mean of the distribution is given by the formula:

$$\textbf{Mean} = \frac{x_1 f_1 + x_2 f_2 + x_3 f_3 + \ldots + x_n f_n}{f_1 + f_2 + f_3 + \ldots + f_n}$$

Thinking Time

Given that nine numbers 16, w, 17, 9, x, 2, y, 7 and z have a mean of 11, find the mean of w, x, y and z.

Example 6

The mean of six numbers is 17. Four of the numbers are 15, 17, 20 and 22.
The remaining two numbers each equals to x.
(a) What is the sum of the six numbers?
(b) Find the value of x.

Solution

(a) Since mean $= \dfrac{\text{Sum of six numbers}}{6}$, then

$$\begin{aligned}\text{Sum of the six numbers} &= \text{mean} \times 6 \\ &= 17 \times 6 \\ &= 102\end{aligned}$$

(b) $$\begin{aligned}\text{Sum of the six numbers } = x + x + 15 + 17 + 20 + 22 &= 102 \\ 2x + 74 &= 102 \\ 2x &= 28 \\ x &= 14\end{aligned}$$

Median

In a survey to find out the living standards of the people in a certain area, the monthly incomes of nine households chosen from the area were obtained:

Household	1	2	3	4	5	6	7	8	9
Income (in dollars)	1720	1940	1960	2030	2050	2250	2400	2550	22 250

The mean income of these nine households is \$4350. This number gives a distorted picture of the standard of living of the people in that area because eight out of nine households have incomes well below \$4350. The 'problem' lies with Household 9 with an extremely high income of \$22 250. To 'get rid' of this extreme value, one way is to use another type of average called the median, which is the middle value after arranging the data in increasing or decreasing order. In this example, the data is already in ascending order and the middle value or median is \$2050. This gives a more accurate picture as most incomes are between \$1720 to \$2550.

Example 7

Find the median of the following set of numbers.

12, 15, 15, 17, 20, 25, 32

Solution

Total number of data (or total frequency) = 7 (odd)

Middle position $= \dfrac{7+1}{2}$

= 4th position

Median = data in the 4th position
= 17

12, 15, 15, (17), 20, 25, 32

This example shows that the median for an **odd** number of data is the middle value when the data are arranged in ascending/descending order.

Your Attention

You can see very clearly from the data in Example 7 that the median is the 4th value. If median position $= \dfrac{\text{total frequency}}{2} = \dfrac{7}{2} = 3.5$, then it is wrong. So median position $= \dfrac{\text{total frequency}+1}{2} = \dfrac{7+1}{2} = 4$.

Example 8

Find the median of the following set of numbers:

12, 15, 15, 17, 20, 25, 32, 32

Solution

Total number of data (or total frequency) = 8 (even)

Middle position $= \dfrac{8+1}{2}$

$= 4.5$th position

Median = mean of the 4th and 5th values

$= \dfrac{17+20}{2}$

$= 18.5$

> **Your Attention**
>
> *You can see very clearly from the data in Example 8 that the median is between the 4th and the 5th values. If median position*
>
> $= \dfrac{\text{total frequency}}{2} = \dfrac{8}{2}$
>
> *= 4, then it is wrong. So median position*
>
> $= \dfrac{\text{total frequency} + 1}{2} = \dfrac{8+1}{2}$
>
> *= 4.5, i.e. the median is the mean of the 4th and 5th values.*

This example shows that the median for an **even** number of data is the mean of the two middle values when the data are arranged in ascending/descending order.

Thinking Time

Does it matter if we arrange the above data in descending order? Why?

It may be obvious where the median lies without finding the median position. But the next three examples will show you why it is helpful to find the median position when the data set is large.

Example 9

Let us revisit Mr Tan's class of 30 students again. Recall in the previous section that the mean score of the test is 33.5 marks. Now Mr Tan is worried that the high scores of a few students in his class (extreme values) might distort the mean score. So he decides to find the median score from the stem and leaf diagram. What is the median score and what does this show?

Stem	*Leaf*
2	0 4 4 5 5 5 6 7 8 9
3	0 0 0 6 6 7 7 7 7 7 7 8 8 8 9 9
4	0 4 4 8

Solution

Total number of data (or total frequency) = 30 (30 students or 30 leaves)

Middle position = $\frac{30 + 1}{2}$

= 15.5th position

Median score = mean of the 15th and 16th scores

$= \frac{36 + 37}{2}$ [find the 15th and 16th scores from the stem and leaf diagram]

= 36.5 marks

Mr Tan finds that there is an equal number of students who score below 36.5 marks and who score above 36.5 marks. Since the median score (36.5 marks) is not very much different from the mean score (33.5 marks), Mr Tan does not need to worry that his set of data is affected by extreme values.

Your Attention

The data in a stem and leaf diagram are already arranged in ascending order.

Example 10

The following table shows the distribution of heights of 15 students. Find the median height.

Height (cm)	*152*	*154*	*156*	*158*	*160*
No. of students	*2*	*1*	*3*	*2*	*7*

Solution

Middle position $= \dfrac{15+1}{2}$

$= 8\text{th position}$

Finding the 8th value from the frequency table, we have

median height = 158 cm.

Don't just take the value shown in the middle row as the median.

Height (cm)	*No. of students*
152	*2*
154	*1*
156	*3*
158	*2*
160	*7*

3rd row

Someone may simply think the median is 156 cm since the 3rd row is in the middle of the 5 rows. This is wrong.

Example 11

The following table shows the distribution of heights for 20 students. Calculate the median height.

Height (cm)	*152*	*154*	*156*	*158*	*160*	*165*
No. of students	*2*	*5*	*3*	*6*	*3*	*1*

Solution

Middle position $= \dfrac{20+1}{2}$

$= 10.5\text{th position}$

Median height = mean height of the 10th and 11th values

$= \dfrac{156+158}{2}$

$= 157 \text{ cm}$

The median age of the Singapore population in 2005 was 36 years old.

In statistics, the mean, median and mode are known as **averages**.

Explorations

Create as many sets of data as you can that satisfy all the following conditions:

Each set consists of 7 data.

The range (maximum value – minimum value) is 10 units.

The mean is greater than the median.

Show that each of the data sets you have created satisfies the above conditions.

Comparison of the Mean, Median and Mode

The **mean** is the most widely used **average**. It is usually preferred over the median and the mode because *all* the values in the data are used in calculating the mean, unlike the median and the mode. The mean is the most reliable measure provided there are no extreme values in the data. When a set of data contains extreme values, the median is a better indicator because it is not affected by extreme values.

The **median** is the prefered **average** for describing economic, sociological and educational data. It is popular in the study of the social sciences because much of the data in the social sciences contain extreme values, as in the set of household incomes we discussed earlier.

The **mode** is an **average** that is more useful in business planning as a measure of popularity that reflects opinion. Examples include the drink seller wanting to know the most popular brand of drinks, manufacturers who want to know the most popular sizes of shoes, shorts, skirts, etc., all of which have been mentioned earlier.

Problem Solving Tips

Suppose there are 25 employees in a company. The distribution of their monthly wages is shown in the table below.

Salary	*Frequency*
$50 000	1
$25 000	1
$15 000	4
$10 000	2
$ 5000	5
$ 1500	12

When asked about the monthly salary of the employee, the following statements are made:
***Employee A** says that each employee earns $7920 on average.*
***Employee B** says that over 50% of the employee earns at least $5000.*
***Employee C** says that almost half of the employees earn $1500.*
These statements seem to contradict one another. Who is right?

The following three examples will illustrate the different uses of mean, median and mode.

Example 12

In a company, the monthly salaries (in dollars) of the employees are:
2000, 2100, 2800, 2400, 2500, 2000, 10000, 2900, 2700, 2600

Find the
(a) mean,
(b) mode, and
(c) median of the monthly salaries.

Which average gives the best picture of how much money employees in the company earn? Why?

Solution

(a) Mean salary $= \frac{\$32\,000}{10}$

$= \$3200$

(b) Mode = \$2000 (occurs twice)

(c) 2000, 2000, 2100, 2400, 2500, 2600, 2700, 2800, 2900, 10 000

Middle position $= \frac{10 + 1}{2}$

$= 5.5\text{th position}$

Median salary $= \frac{\$2500 + \$2600}{2}$

$= \$2550$

The mean is \$3200 but nine of the ten employees earn less than \$3000.

The mode is \$2000 but eight of the ten employees earn more than \$2000.

Therefore, the median of \$2550 gives the best picture in this case because nine of the ten employees earn between \$2000 and \$2900 (inclusive) and it is not affected by the extreme value, \$10 000.

Example 13

The owner of a shop kept a record of the sizes of blouses sold on a particular day. The record showed:

8, 8, 10, 8, 10, 12, 10, 8, 8, 12

Find the (a) mean,
(b) mode, and
(c) median of the sizes of blouses sold on that day.

Which average gives the best picture of the size of blouses sold on that day? Why?

Solution

(a) Mean size $= \frac{94}{10}$

$= 9.4$

(b) Mode = 8 (occurs five times)

(c) 8, 8, 8, 8, 8, 10, 10, 10, 12, 12

Middle position $= \frac{10 + 1}{2}$

$= 5.5$th position

Median size $= \frac{8 + 10}{2}$

$= 9$

The mean is 9.4 but there is no such size. The median is 9 but there is no such size. Even if there is such a size, it does not tell you much.

Therefore, the mode of 8 gives the best picture in this case because it shows the most popular size sold and the shop owner will be interested to bring in more blouses of size 8.

Example 14

A survey was conducted among 20 families to find out the number of children they have. The table of data collected is as follows:

No. of children	*1*	*2*	*3*	*4*
No. of families	*1*	*11*	*6*	*2*

Find the
(a) mean,
(b) mode, and
(c) median of the number of children in a family.

Which average gives the best picture of the number of children in a family?
What does it mean when the average number of children is not a whole number?

Solution

(a) $\text{Mean} = \dfrac{1 \times 1 + 2 \times 11 + 3 \times 6 + 4 \times 2}{20}$

$= \dfrac{49}{20}$

$= 2.45$

(b) Mode = 2

(c) $\text{Middle position} = \dfrac{20 + 1}{2}$

$= 10.5\text{th position}$

$\text{Median} = \dfrac{2 + 2}{2}$

$= 2$

The mean of 2.45 children in a family gives the best picture in this case because all the values in the data are used in calculating the mean. A mean of 2.45 does **not** mean that every family has 2.45 children. It just means that there are 245 children in 100 families, or in this case, 49 children in 20 families. If we were to use the mode or median of 2, it will mean that there are 40 children in 20 families, when in fact, there are 49 children in 20 families.

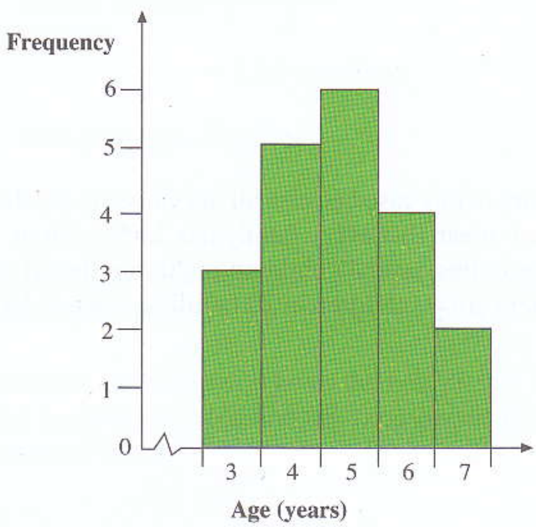
JOURNAL WRITING

1. Find out the prices of 5 different brands of cigarettes available in the market and the number of cigarettes in each packet.
2. Peter's father smokes an average of 30 cigarettes per day.
   (a) Using the mean price of the 5 brands of cigarettes in question 1, calculate how much Peter's father can save if he gives up smoking for 1 year.
   (b) Which of the following items do you think Peter's father can buy if he gives up smoking for 1 year?
      (i) Digital camera
      (ii) Nicam stereo TV
      (iii) Mini hi-fi system
      (iv) Refrigerator
      (v) All of the above items
3. Is kicking the habit of smoking a good idea? Write an article about your findings and share it with your classmates.

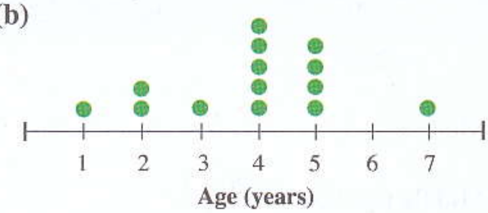
Exercise 11b

1. Find the mode of the following set of data:
(a) 2, 5, 8, 3, 7, 1, 5, 3, 9, 7, 3
(b) 8.1, 7.7, 7.8, 9.3, 6.4, 7.7, 9.3, 5.8, 7.7, 8.1
(c) 12, 17, 16, 14, 13, 16, 11, 14

2. Find the mode of the following set of data:
(a)



(b)



(c)

Stem	Leaf
2	1 1 2 3 3
3	7 8 9 9
4	2 3 5 5 7 7
5	3 3 7
6	0 0 0 1

(d)

x	0	1	2	3	4	5
Freq	2	0	3	4	1	2

3. The number of bus passengers travelling on a certain route was recorded as shown below.

29, 42, 45, 39, 41, 38, 37, 38, 43, 40, 36, 35, 32, 38

Find the mean number of passengers.

4. The prices (in \$) of various computer books on designing of web pages in a bookshop are given below.

19.90, 24.45, 34.65, 26.50, 44.05, 38.95, 56.40, 48.75, 29.30, 35.65

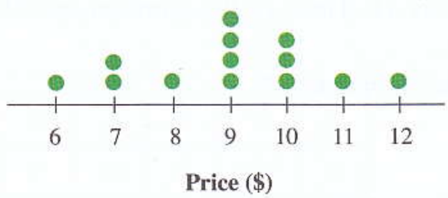
Find the mean price of these books.

5. Find the mean of the following set of data:

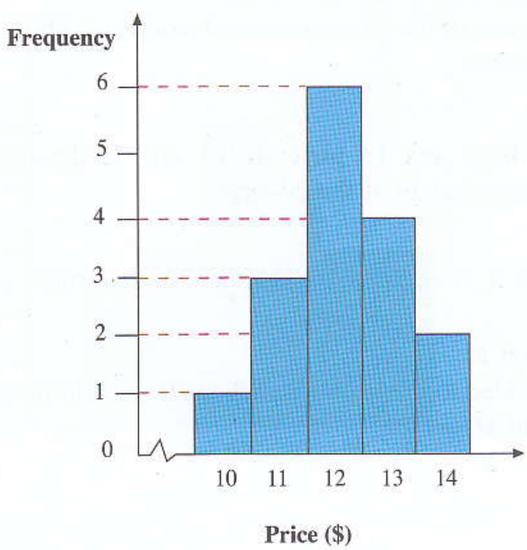
(a)

Stem	**Leaf**
7	2 3 5 5
8	2 7 8 8 9 9 9
9	1 3 7 7
10	2 7 8

(b)



(c)



6. In a soccer season, a soccer team played 30 matches. The table below shows the distribution of the number of goals scored per match.

No. of goals scored per match	0	1	2	3	4	5	6
No. of matches	6	8	5	6	2	2	1

Calculate the mean number of goals scored per match.

7. The table below shows the marks obtained by 40 pupils in English and Mathematics tests.

Marks	1	2	3	4	5	6	7	8	9	10
No. of candidates (English)	1	6	14	4	8	2	4	0	1	0
No. of candidates (Mathematics)	4	1	6	5	10	3	5	3	1	2

Calculate the mean marks for each subject.

8. The mean of six numbers is 41. Three of the numbers are 32, 31 and 42. The remaining three numbers each equals to a.

(a) What is the sum of the six numbers?

(b) Find the value of a.

9. A small factory employs 7 experienced and 5 inexperienced workers. The mean monthly wage of these 12 workers is \$700.

(a) Calculate the total wage bill for the 12 workers.

(b) Given that the mean wage of the 5 inexperienced workers is \$602, calculate the mean wage of the 7 experienced workers.

10. The mean height of 20 boys and 14 girls is 161 cm. If the mean height of the 14 girls is 151 cm, calculate the mean height of the 20 boys.

11. The heights of three plants A, B and C in a garden are in the ratio 2 : 3 : 5. Their mean height is 30 cm.

(a) Find the height of plant B.

(b) If another plant D is added to the garden and the mean height of the four plants is now 33 cm, find the height of plant D.

12. The mean of three numbers x, y and z is 6 and the mean of five numbers x, y, z, a and b is 8. Find the mean of a and b.

13. Find the median of the following set of data:

(a) 7, 6, 4, 8, 2, 5, 11

(b) 13, 10, 10, 14, 15, 18, 11, 100

(c) 22, 27, 18, 16, 33, 31

14. Find the median of the following set of data:

(a)

Stem	Leaf
3	0 1 4
4	2 5 8 9
5	1 7 8 9 9 9
6	3 7 7
7	4 6

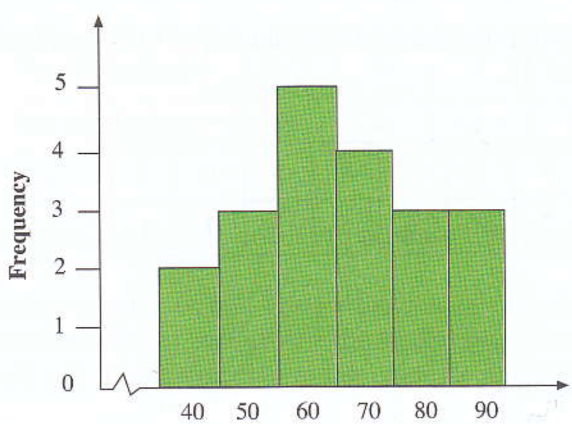
(b)

x	30	35	40	45	50	55	60
Frequency	5	6	10	8	7	5	2

(c)

x	0	1	2	3	4
Frequency	5	10	8	4	3

(d)



15. Determine the mean, median and mode of the following sets of numbers.

(a)	8	11	14	13	14	9	15			
(b)	2	5	6	3	7	8	4	12		
	11	9	10	7	6	8	9	7		
(c)	88	93	85	98	102	98	93	104	102	98

(d)

x	0	1	2	3	4	5	6
Frequency	1	2	3	4	5	6	7

16. The following table shows the monthly wages of 27 employees in a certain factory in 2008.

Wages \$(x)	670	760	850	960	1000	1200
No. of employees (f)	4	9	8	3	2	1

Find **(a)** the mean monthly wage,
(b) the median monthly wage,
(c) the modal monthly wage.

17. Two dice are tossed 30 times. The sum of the scores each time is shown below:

Score (x)	2	3	4	5	6	7	8	9	10	11	12
Frequency (f)	1	1	3	4	6	8	3	2	1	1	0

Find the mean, the median and the mode of the scores.

18. The table below shows the frequency distribution of the number of spelling mistakes in a composition made by each pupil in a class of 36.

No. of mistakes (x)	0	1	2	3	4	5	6	7
No. of pupils (f)	3	7	10	6	5	3	1	1

Find **(a)** the mean,
(b) the median,
(c) the mode of the distribution.

19. **(a)** The median of a set of eight numbers is $4\frac{1}{2}$. Given that seven of the numbers are 9, 2, 3, 4, 12, 13 and 1, find the eighth number.

(b) The mean of a set of six numbers is 2 and the mean of another set of ten numbers is m. If the mean of the combined set of sixteen numbers is 7, find the value of m.

20. Peter and Paul were playing golf. The scores on the first nine holes are shown in the table below. In golf, the **lower** the score, the **better**.

Hole	1	2	3	4	5	6	7	8	9	***Total***
Peter's score	3	2	5	7	3	2	2	4	17	**45**
Paul's score	4	4	6	8	3	3	2	6	6	**42**

On the ninth hole, Peter got stuck in a sand trap and lost the game.

(a) Calculate the mean score on the nine holes for each player.

(b) Which player did better on most of the holes? Do the mean scores indicate this?

(c) What were the median scores for both players?

(d) Find the mode of each player's scores.

(e) Which average – the mean, the median or the mode – do you think gives the best comparison of the abilities of Peter and Paul? Why?

21. Two classes, each with 21 students were given a physical fitness test. The results of the number of pull-ups performed in 30 seconds were recorded in the tables below:

Class A	**No. of pull-ups**	$\leqslant 5$	6	7	8	9	$\geqslant 10$
	No. of students	3	7	4	4	2	1

Class A	**No. of pull-ups**	$\leqslant 5$	6	7	8	9	$\geqslant 10$
	No. of students	3	4	4	7	2	1

(a) Can you calculate the mean number of pull-ups performed by the students in each class? Why?

(b) Find the median number of pull-ups for the students in each class.

(c) Find the modal number of pull-ups for the students in each class.

(d) Which would you use, the median or the mode, to compare the results of the two classes? Why?

22. There are three different netball teams and each team has played five games. The following table shows the scores of each team.

	Game 1	Game 2	Game 3	Game 4	Game 5
Cheetah	65	95	32	101	88
Puma	50	90	65	87	87
Jaguar	90	85	46	44	80

(a) Suppose you want to join one of the three netball teams. You want to join the one that is doing the best so far. If you rank each team by their mean scores, which team would you join?
(b) If, instead of using mean scores, you decide to use the median score of each team to make your decision, which team would you join?
(c) Suppose you are the coach of the team Puma and you are being interviewed about your team for the local newspaper. Would it be better for you to report your mean score or your median score? Why?

23. Your friends and you are comparing the number of storybooks that have been read in the past year. The following table illustrates the number of books each person read in each month.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Amy	2	3	2	3	1	4	2	3	1	1	2	2
Bruce	3	1	2	2	3	5	4	1	2	2	3	1
Carol	2	1	4	2	2	3	3	1	1	1	2	1
Danny	1	2	3	2	4	1	3	2	5	2	3	1

(a) By comparing their modes, which person read the least number of books per month?
(b) By comparing their medians, which person read the most number of books per month?
(c) By comparing their means, rank these four people in the order of most number of books read to least number of books read.

Averages for Grouped Data

We have learnt how to group data in Book 1. Now we will learn how to find the mean of grouped data.

Example 15

The table below shows the number of a popular magazine sold to customers in different shops in the month of July.

Number of Magazines sold	*70 – 74*	*75 – 79*	*80 – 84*	*85 – 89*	*90 – 94*	*95 – 99*	*100 –104*
Frequency	*4*	*11*	*15*	*24*	*18*	*9*	*3*

Estimate the mean of the distribution.

Solution

We can only **estimate** the mean of the distribution because we only know there are 4 shops which sold 70 – 74 magazines but we do not know the exact number of magazines sold in each shop unless we have the original set of data.

To calculate the mean of a set of grouped data, we need to represent all the data in each of the class intervals by a value. For example, which value best represents the class interval 70 – 74? We cannot use 70 because most of the data could be more than 70. We cannot use 74 because most of the data could be less than 74. So a good value to represent the class interval 70 – 74 is the mid-value 72. The mid value is obtained by $\frac{70 + 74}{2} = 72$.

Class interval	Mid-value (x)	Frequency (f)	fx
70 – 74	72	4	288
75 – 79	77	11	847
80 – 84	82	15	1230
85 – 89	87	24	2088
90 – 94	92	18	1656
95 – 99	97	9	873
100 – 104	102	3	306
		$\sum f = 84$	$\sum fx = 7288$

Therefore, the mean number of magazines sold is $\bar{x} = \frac{\sum fx}{\sum f} = \frac{7288}{84} = 86.8$ (correct to 3 sig. fig.)

In general, the mean of a set of grouped data is $\bar{x} = \frac{\sum fx}{\sum f}$, where x is the mid-value of the class interval, and f is the frequency of the class interval.

Using Calculator to Find Mean

We can also use the statistical functions of a calculator to find the mean of a set of data directly.

Now, let us find the mean of the following set of numbers using a calculator.

10 , 15 , 15 , 20 , 20 , 20 , 26 , 30

Step 1 Press the key [MODE] and select [1] for statistics.

Step 2 Press [0] for single variable statistics.

Your Attention

The example shown here is valid only for calculators of certain models. For calculators of other model, you may need to refer to the user's manual.

Step 3 Key in the data as shown below:

[1] [0] [DATA]

[1] [5] [DATA]

[DATA] [For repeat data, just key in [DATA] without the number.]

[2] [0] [DATA]

[DATA]

[DATA]

[2] [6] [DATA]

[3] [0] [DATA]

Step 4 Press the keys [RCL] and [n]. The screen will then show the number of data keyed in: $n = 8$

Press the keys [RCL] and [$\bar{x}$]. The screen will then show the mean of the set of data: $\bar{x} = 19.5$

For grouped data, we can also calculate the mean directly using a scientific calculator. Let's use the calculator to work out Example 15.

Step 1 Press the key [MODE] and select [1] for statistics.

Step 2 Press [0] for single variable statistics.

Step 3 Key in the data as shown below:

[7] [2] [(x, y)] [4] [DATA]

[7] [7] [(x, y)] [1] [1] [DATA]

[8] [2] [(x, y)] [1] [5] [DATA]

[8] [7] [(x, y)] [2] [4] [DATA]

[9] [2] [(x, y)] [1] [8] [DATA]

[9] [7] [(x, y)] [9] [DATA]

[1] [0] [2] [(x, y)] [3] [DATA]

Step 4 Press the keys [RCL] and [n]. The screen will then show the number of data keyed in: $n = 84$

Press the keys [RCL] and [$\bar{x}$]. The screen will then show the mean: $\bar{x} = 86.76190476$

1. The masses of 100 pebbles (in grams) picked up by a boy from a beach are as follows:

Mass of pebbles (x g)	Frequency
$55 < x \leqslant 65$	2
$65 < x \leqslant 75$	3
$75 < x \leqslant 85$	9
$85 < x \leqslant 95$	23
$95 < x \leqslant 105$	26
$105 < x \leqslant 115$	21
$115 < x \leqslant 125$	10
$125 < x \leqslant 135$	5
$135 < x \leqslant 145$	1

Copy and complete the table below and hence, find the mean mass of the 100 pebbles.

Mass of pebbles (g)	Mid-point (x)	Frequency (f)	fx
$55 < x \leqslant 65$	60	2	120
$65 < x \leqslant 75$			
$75 < x \leqslant 85$			
$85 < x \leqslant 95$			
$95 < x \leqslant 105$			
$105 < x \leqslant 115$			
$115 < x \leqslant 125$			
$125 < x \leqslant 135$			
$135 < x \leqslant 145$			
		$\sum f =$	$\sum fx =$

2. The heights of 40 plants were measured correct to the nearest centimetre. Copy and complete the table below and hence, calculate the mean height of these 40 plants.

Heights (cm)	Mid-point (x)	Frequency (f)	fx
1–10	5.5	4	
11–20		6	
21–30		14	
31–40		6	
41–50		10	
		$\sum f =$	$\sum fx =$

3. Thirty bulbs were life-tested and their lifespans to the nearest hour are as follows:

167 171 179 167 171 165 175 179 169 168 171 177 169 171 177

173 165 175 167 174 177 172 164 175 179 179 174 174 168 171

(a) Find the mean lifespan by dividing their sum by 30.

(b) Find the mean lifespan by grouping the lifespans using the class intervals 164 – 166, 167–169 and so on.

Lifespan (h)	Tally	Mid-value (x)	Frequency (f)	fx
164 – 166		165		
167 – 169				
170 – 172				
173 – 175				
176 – 178				
179 – 181				
			$\sum f =$	$\sum fx =$

(c) Repeat **(b)** using the class intervals 164 – 165, 166 – 167 and so on. Leave your answer to 4 significant figures.

Lifespan (h)	Tally	Mid-value (x)	Frequency (f)	fx
164 – 166		164.5		
167– 169				
170 – 171				
172 – 173				
174 – 175				
176 – 177				
178 – 179				
			$\Sigma f =$	$\Sigma fx =$

(d) Are the values of the mean of the distribution found in **(a)**, **(b)** and **(c)** the same? Which is the actual mean? What do the answers for **(b)** and **(c)** tell you?

4. The following table shows the time taken for 100 lorries to travel between two towns using a certain route.

Time taken (t min)	Number of lorries
$116 < t \leqslant 118$	1
$118 < t \leqslant 120$	6
$120 < t \leqslant 122$	23
$122 < t \leqslant 124$	28
$124 < t \leqslant 126$	27
$126 < t \leqslant 128$	9
$128 < t \leqslant 130$	5
$130 < t \leqslant 132$	1

Calculate the mean travelling time.

5. The table below shows the distribution of ages of the members of a club.

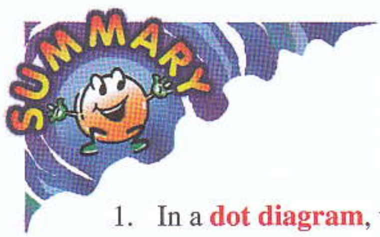
Age (years)	20 – 24	25 – 29	30 – 34	35 – 39	40 – 44	45 – 49	50 – 54
Frequency	22	48	60	36	22	10	2

Calculate the mean age of the members of the club by using the statistical functions of a scientific calculator.

6. A machine in a factory broke down 100 times in a certain year. The length of time taken to repair the machine each time was recorded. The table below shows the distribution of the lengths of time (t minutes) taken to repair the machine

Repair time	Frequency
$0 < t \leqslant 10$	3
$10 < t \leqslant 20$	13
$20 < t \leqslant 30$	30
$30 < t \leqslant 40$	25
$40 < t \leqslant 50$	14
$50 < t \leqslant 60$	8
$60 < t \leqslant 70$	4
$70 < t \leqslant 80$	2
$80 < t \leqslant 90$	1

Calculate the mean length of time taken to repair the machine by using the statistical functions of a scientific calculator.



1. In a **dot diagram**, values are presented by **dots above a horizontal number line.**
2. In a **stem and leaf diagram**, a **value** is **split into** two parts, namely **a stem and a leaf**.
3. A set of data can be described by numerical quantities called **averages**.
4. The three common averages are the **mean**, the **median** and the **mode.**
5. The **mode** is the number that **occurs most frequently**.
6. The **mean** is the **sum of values divided by** the **number of values** in a set of data.
7. The **median** for an **odd** number of data **is the middle value** when the data are arranged in ascending/descending order. The **median** for an **even** number of data **is the mean of the two middle values** when the data are arranged in ascending/descending order.
8. The mean of a set of grouped data is $\bar{x} = \frac{\sum fx}{\sum f}$, where x is the mid-value of the class interval, and f is the frequency of the class interval.

Review Examples 11

Example 1

The scores of a Mathematics test, marked out of a total of 60, taken by two classes A and B are given as follows:

Class A		Class B
Leaf	Stem	Leaf
5	1	0 2 7 7 8
9 3	2	
4 3 2	3	
9 8 7 6 6 5 5	4	0 0 0 0 1 1 2 2 2 3 8 8 8 8 8
6 5 4 4 2 1 0	5	

(a) How many students are there in each class?
(b) Comment on the distribution of each class.
(c) How would you judge which class has performed better?

Solution

(a) There are 20 students in each class.

(b) The test scores in Class A are well spread but the test scores in Class B fall within a narrow range.

(c) By calculating the mean, we are able to judge which class has performed better.

Mean score of Class A = $\frac{864}{20} = 43.2$

Mean score of Class B = $\frac{725}{20} = 36.25$

$\therefore$ Class A has performed better.

Example 2

The table below shows the frequency distribution of the number of spelling mistakes made by each pupil in an essay.

Number of mistakes	*0*	*1*	*2*	*3*	*4*	*5*
Number of pupils	*4*	*8*	*x*	*6*	*5*	*4*

(a) If the median is 3, what are the possible values of x?
(b) Find x if the mean mistake is 2.4.

Solution

(a) Write the data as follows:

$$\underbrace{0, ..., 0}_{4}, \underbrace{1, ..., 1}_{8}, \underbrace{2, ..., 2}_{x}, \underbrace{3, ..., 3}_{6}, \underbrace{4, ..., 4}_{5}, \underbrace{5, ..., 5}_{4}$$

The biggest value of x occurs when the median is here
$\therefore 4 + 8 + x = 5 + 5 + 4$
$x = 2$
$\therefore$ the biggest value of $x = 2$
$\therefore$ the possible values of x are 0, 1, 2.

The smallest value of x occurs when the median is here (if possible)
$\therefore 4 + 8 + x + 5 = 5 + 4$
$x = -8$ (not possible)
$\therefore$ the smallest value of $x = 0$
(no need to determine exact position of median)

(b) $$\frac{0 \times 4 + 1 \times 8 + 2 \times x + 3 \times 6 + 4 \times 5 + 5 \times 4}{4 + 8 + x + 6 + 5 + 4} = 2.4$$

$$\frac{66 + 2x}{27 + x} = 2.4$$

$$66 + 2x = 2.4\,(27 + x)$$

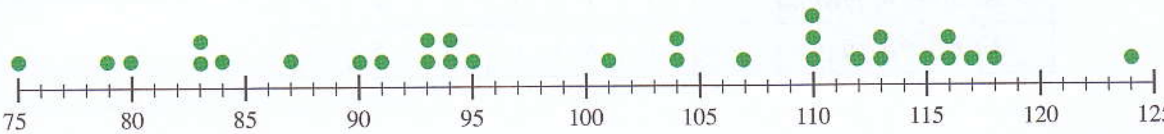
$$66 + 2x = 64.8 = 2.4x$$

$$0.4x = 1.2$$

$$x = 3$$

Review Questions 11

1. The following diagram represents the scores of 30 students in a quiz:



(a) What is the most common score?
(b) There is an exceptionally high score. Identify this score.
(c) Comment briefly on what the data indicate.

2. The following data represent the masses, in grams, of 50 pencil sharpeners:

Stem (Whole Number)	Leaf (1st Decimal Place)
5	2
6	8 9 9
7	0 0 0 1 2 2 2 2 2 2 3 3 4 4 4 5 6 6 7 7 7 9
8	0 1 3 4 4 5 5 6 6 6
9	0 1 1 2 2 3 4 4 5 8
10	2 4 5
11	9

(a) Find the most common mass.
(b) Comment briefly what the data indicate.

3. (a) The distribution of 300 values of a variable x is shown in the following table:

x	0	1	2	3	4	5	6	7
Frequency	18	37	69	66	45	26	24	15

For this distribution, find

(a) the mode,
(b) the median,
(c) the mean.

4. A box contained five cards numbered 1, 2, 3, 4 and 5. A card was drawn from the box, its number noted and then replaced. The process was repeated 100 times and the table below shows the resulting frequency distribution.

Card	1	2	3	4	5
Frequency	21	x	y	18	17

(a) Show that $x + y = 44$.
(b) If the mean of the distribution is 2.9, show that $2x + 3y = 112$.
(c) From **(a)** and **(b)**, find the value of x and of y, and then state the mode and the median of the distribution.

5. A bag contained six cards, each bearing one of the numbers 1, 2, 3, 4, 5, 6. A card was drawn from the bag, its number noted down and then replaced. This was repeated 60 times and the frequency distribution table below shows the results.
Given that the mean of this distribution is 3.6,

(a) find the values of m and n,
(b) state the mode and median of the distribution.

Number	1	2	3	4	5	6
Frequency	m	14	n	5	10	11

6. The table below shows the number of pupils in a class of 36 who scored marks between 3 and 9 inclusive in a test.

Marks	3	4	5	6	7	8	9
Number of pupils	3	4	x	7	y	5	4

(a) If the mean is 6, calculate the values of x and y.
(b) With the values of x and y, state the modal mark and the median mark.

7. The numbers 24, 22, 34, 28, 29, 24, 25, 29, x and y have a mode of 29 and a median of 27. Find the values of x and y, given that $x < y$.

8. The table below shows the number of goals scored by soccer teams in a competition.

Number of goals	0	1	2	3	4	5
Number of teams	15	x	6	5	5	0

(a) Find the value of x if 40 teams took part in the competition.
(b) State the biggest possible value of x if the mode is 0 goal.
(c) Write down the smallest possible value of x if the median is 1 goal.

9. In an examination, each pupil in a group scores either 5, 10 or 15 marks. The number of pupils scoring each mark is shown in the table below:

Mark	5	10	15
Number of pupils	8	12	x

(a) If the mode is 10, write down the range of values of x.
(b) If the median mark is 10, write down the largest possible value of x.
(c) Using the value of x found in **(b)**, calculate the mean mark.

10. Some children were asked the number of books they read in a week. Their feeback are shown in the table below:

Number of book read	0	1	2	3
Number of children	5	6	3	x

(a) Write down the greatest possible value of x given that the median is 1.
(b) Write down the greatest possible value of x given that the mode is 1.
(c) Calculate the value of x given that the mean is 1.

11. Some women were asked the number of magazines that they read in a week. The table below shows the results.

Number of magazines read	0	1	2	3
Number of women	5	2	1	x

(a) If the median is 2, find the value of x.
(b) Find the greatest possible value of x if the median is 1.

12. The heights of a group of 30 children were measured to the nearest centimetre and the readings are recorded as shown below:

122	144	136	136	140	139	126	120	125	129	127	116	132	138	124
135	122	137	135	129	133	130	128	118	131	127	128	147	133	119

(a) Copy and complete the table given.
(b) Estimate the mean height of the 30 children using the table below.
(c) Estimate the percentage of children whose heights are below 135 cm.

Height (cm)	**Mid-value (x)**	**Frequency (f)**	fx
115 – 119			
120 – 124			
125 – 129			
130 – 134			
135 – 139			
140 – 144			
145 – 149			
		$\Sigma f =$	$\Sigma fx =$

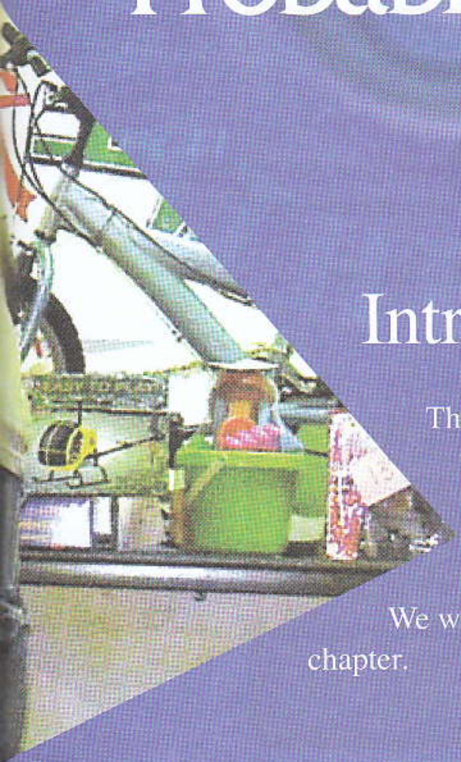
In this chapter, you will learn how to

- ***measure chance using probability;***
- ***list all the possible outcomes in a simple chance situation;***
- ***find the probability of single events.***

C H A P T E R

12

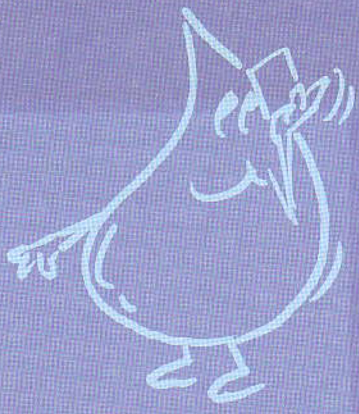
Probability



Introduction

There are 20 prizes for the 400 residents who attended the Christmas party organised by the Resident Committee in one of our housing estates. What is the probability that a resident will be able to get a prize during the draw?

We will learn how to calculate the probability of such events in this chapter.



Introduction

We often make statements such as:

(a) " There is a 50:50 **chance** of our school winning the National School Basketball Championship."

(b) "I cannot **predict** whether I will get a 'six' in my next throw of a die."

(c) "It will **probably** rain today."

We make such statements because we are uncertain whether an **event** will occur or not. For an uncertain event, we can talk about its chance of occurring.

Let's consider the chance that each of the following events will occur:

(e) It will rain in Singapore tomorrow.
(f) It will snow in Singapore tomorrow.

Our experience tells us that it often rains in Singapore, but it has never snowed here. So the chance that event (e) will occur is quite high, but the chance that event (f) will occur is extremely small.

The measure of chance is known as **probability**, which we will learn in this chapter.

Do it Yourself

Consider some events involving chances of occurring. For each event, make a statement about its chance of occurring.

Experiments and Sample Space

When we perform a scientific experiment, we will get a certain result or outcome. But in probability, the result or outcome is not certain: it depends on chance. The following are some examples of probability experiments and their possible outcomes.

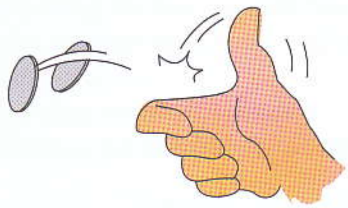
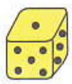
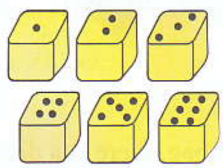
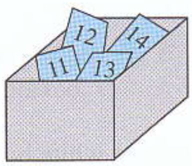
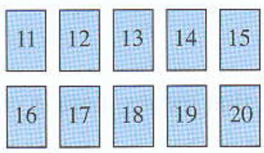
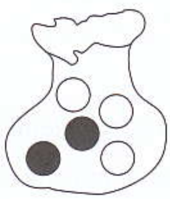
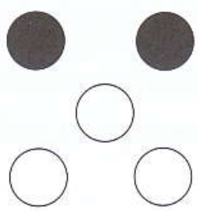
Probability Experiment	Possible Outcomes
1.  Tossing a coin	 Head Tail
2.  Tossing a die	
3.  Ten identical cards numbered 11, 12, 13,..., 20 are placed in a box. One card is drawn at random.	
4.  Three white balls and two black balls of the same size are placed in a bag. One ball is picked from the bag at random.	

Fig. 12.1

The collection of all possible outcomes of an experiment is called the sample space or probability space. It is usually denoted by S and we put all the outcomes in braces { }.

For example, the sample space of each of the four experiments above is :

Experiment 1: $S = \{H, T\}$ where H represents the outcome of getting a head, and T the outcome of getting a tail.

Experiment 2: $S = \{1, 2, 3, 4, 5, 6\}$

Experiment 3: $S = \{11, 12, 13, ..., 20\}$

Experiment 4: $S = \{B_1, B_2, W_1, W_2, W_3\}$ where B represents the outcome of getting a black ball, and W represents the outcome of getting a white ball.

As there are five balls in the bag, there is a difference whether you draw the first black ball or the second black ball. Therefore, we must differentiate between the two black balls by writing them as B_1 and B_2. Similarly, for the three white balls, we write them as W_1, W_2, and W_3.

If we write $S = \{B, W\}$, that means there is only one black ball and one white ball. Then we have **equal chances** of drawing a black ball or a white ball. But if $S = \{B_1, B_2, W_1, W_2, W_3\}$, then we have a higher chance of drawing a white ball because there are more white balls than black balls.

The total number of possible outcomes is the same as the total number of elements in the sample space. It is denoted by $n(S)$. For experiment 1, $n(S) = 2$. What is the value of $n(S)$ for experiments 2, 3 and 4?

Example 1

For each of the experiments below, write down its sample space S and the total number of possible outcomes n(S).

(a) Drawing a card at random from a box containing five identical cards numbered 21, 22, 23, 24 and 25.
(b) Drawing a ball at random from a bag containing 4 identical blue balls and 3 identical red balls.
(c) Selecting a point at random from a circular board coloured as shown in Fig. 12.2(a). We assume the point will never fall exactly on any of the lines separating the four coloured sectors.
(d) Spinning the spinner shown in Fig. 12.2(b) at random (same assumption as (c)).
(e) Choosing a 2-digit number at random.
(f) Tossing a standard tetrahedral die as shown in Fig. 12.2(c).

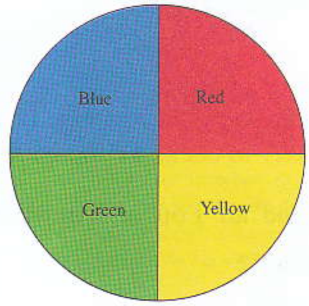


Fig. 12.2(a)

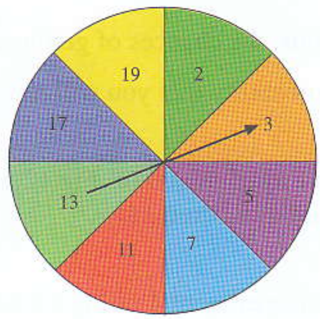


Fig. 12.2(b)


Fig. 12.2(c)

Solution

(a) The sample space is $S = \{21, 22, 23, 24, 25\}$.
$\therefore$ $n(S) = 5$.

(b) The sample space is $S = \{B_1, B_2, B_3, B_4, R_1, R_2, R_3\}$ where B represents blue and R represents red.
$\therefore n(S) = 7$

(c) The sample space is S = {Blue, Red, Green, Yellow}.
$\therefore$ $n(S) = 4$.

(d) The sample space is $S = \{2, 3, 5, 7, 11, 13, 17, 19\}$.
$\therefore$ $n(S) = 8$.

(e) The sample space is $S = \{10, 11, 12, ..., 99\}$.
$\therefore$ $n(S) = 99 - 10 + 1 = 90$.

(f) A tetrahedral die is a 4-sided die.
Unless stated otherwise, we always assume that the numbers start from 1 for a standard tetrahedral die.
So the sample space is $S = \{1, 2, 3, 4\}$.
$\therefore$ $n(S) = 4$.

Your Attention

There are 11 numbers from 20 to 30. If you take $30 - 20 = 10$, you end up with only 10 numbers. So to find how many numbers from 20 to 30, you need to take $30 - 20 + 1 = 11$. Similarly, there are $99 - 10 + 1 = 90$ numbers from 10 to 99. How many numbers are there from 60 to 77?

Definition of Probability

In Experiment 1 of Fig. 12.1, if the coin is **fair** or **unbiased**, then each of the two outcomes is **equally likely** to occur, i.e there are equal chances of getting a 'head' or a 'tail'. Thus, the chances of getting a 'head' are 1 out of 2, or we say the probability of getting a 'head' is $\frac{1}{2}$. What is the probability of getting a 'tail'?

In Experiment 2 of Fig.12.1, if the die is fair, then each of the six outcomes is equally likely to occur. Thus, the chances of getting a 'six' are 1 out of 6, or the probability of getting a 'six' is $\frac{1}{6}$. What is the probability that you will not get a 'six'?

In Experiment 3 of Fig.12.1 , if the card is drawn at random, then what is the probability of picking a card numbered 14?

In Experiment 4 of Fig.12.1, what is the probability of drawing a white ball?

Let's denote this event by E. In Experiment 3 of Fig.12.1, suppose we are interested in the event "picking a prime number". Then there are four outcomes: 11, 13, 17, 19 that favour the occurrence of the event E. This means that if you pick any of these four favourable outcomes, then the event E will happen.

So the chances of the event E occurring are 4 out of 10, i.e. the probability of the event E happening, written as $P(E)$, is:

$$P(E) = \frac{4}{10} = \frac{2}{5}$$

In general, in a probability experiment with m equally likely outcomes, if k of these outcomes favour occurrence of an event E, then the probability of event E happening is:

$$\mathbf{P(E) = \frac{Number\ of\ Favourable\ Outcomes\ for\ Event\ \mathit{E}}{Number\ of\ Possible\ Outcomes} = \frac{n(\mathit{E})}{n(\mathit{S})} = \frac{\mathit{k}}{\mathit{m}}}$$

where $n(E)$ is the number of favourable outcomes in event E and $n(S)$ is the total number of possible outcomes in the sample space S.

Example 2

A card is drawn at random from a box containing 12 identical cards numbered 1, 2, 3, ..., 12.

(a) *Write down the sample space S and the total number of possible outcomes n(S).*

(b) *What is the probability of drawing*
   (i) *a '7'*
   (ii) *an even number*
   (iii) *a prime number*
   (iv) *a perfect square*
   (v) *a negative number*

> FOR YOUR Information
>
> *A prime number is a positive integer that has exactly 2 different positive factors. So 1 is not a prime number.*

Solution

(a) The sample space is $S = \{1, 2, 3, \ldots.., 12\}$.
The total number of possible outcomes is $n(S) = 12$.

(b) (i) Probability of drawing a '7' = $\frac{1}{12}$

or simply P (drawing a '7') = $\frac{1}{12}$.

(ii) There are 6 even numbers from 1 to 12 : 2, 4, 6, 8, 10, 12.

$\therefore$ P(drawing an even number) $= \frac{6}{12}$

$= \frac{1}{2}$.

(iii) There are 5 prime numbers from 1 to 12 : 2, 3, 5, 7, 11.

$\therefore$ P(drawing a prime number) $= \frac{5}{12}$.

(iv) There are 3 perfect squares from 1 to 12 : 1, 4, 9.

$\therefore$ P(drawing a perfect square) $= \frac{3}{12} = \frac{1}{4}$

(v) There are no negative numbers from 1 to 12.

$\therefore$ P(drawing a negative number) $= \frac{0}{12}$.

= 0 (impossible to happen).

Thinking Time

(a) What is the probability of an event that will definitely occur?
(b) What is the probability of an event that will never happen?
(c) Is it possible to have a probability less than 0 or greater than 1?

Example 3

A spinner in the form of a regular hexagon is constructed as shown in Fig. 12.3. When the pointer is spun, what is the probability that the pointer will stop at

(a) 6,
(b) an odd-numbered triangle,
(c) a triangle whose number is a multiple of 3,
(d) a triangle whose number is less than 6,
(e) a triangle whose number is greater than 6,
(f) a triangle whose number is less than 7?

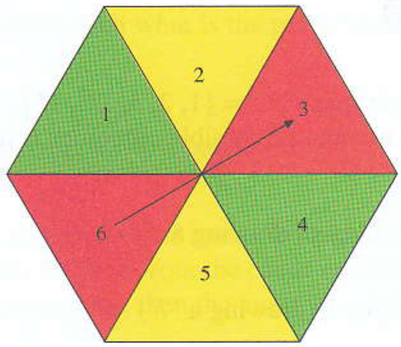


Fig. 12.3

Solution

We assume the pointer will never stop exactly on any of the lines separating the numbers. Since all the triangles have the same angle at the centre of the hexagon, it is equally likely for the pointer to stop at any one of the six triangles.

(a) P (pointer will stop at 6) = $\frac{1}{6}$

(b) There are 3 odd-numbered triangles : 1, 3, 5.

$\therefore$ P (pointer will stop at an odd-numbered triangle) = $\frac{3}{6} = \frac{1}{2}$

(c) There are 2 triangles whose numbers are multiples of 3 : 3, 6.

$\therefore$ P (pointer will stop at a triangle whose number is a multiple of 3) = $\frac{2}{6} = \frac{1}{3}$

(d) There are 5 triangles whose numbers are less than 6 : 1, 2, 3, 4, 5

$\therefore$ P (pointer will stop at a triangle whose number is a less than 6) = $\frac{5}{6}$

(e) There is no triangle whose number is greater than 6.

$\therefore$ P (pointer will stop at a triangle whose number is greater than 6) = $\frac{0}{6} = 0$.

(f) There are 6 triangles whose numbers are less than 7.

$\therefore$ P (pointer will stop at a triangle whose number is less than 7) = $\frac{6}{6} = 1$.

Example 4

In a class of 40 students, 8 are short-sighted. If a student is selected at random, what is the probability that the selected student will be short-sighted?

Solution

P (student selected will be short-sighted) = $\frac{8}{40} = \frac{1}{5}$

Example 5

A card is drawn at random from a pack of 52 playing cards.

(a) What is the total number of possible outcomes of this experiment?
(b) What is the probability of drawing
(i) a black card,
(ii) a red ace,
(iii) a diamond,
(iv) a card which is not a diamond?

Notes

There are 4 suits in a pack of 52 playing cards: spade, heart, club and diamond. Each suit has 13 cards : 1(or ace), 2, 3, ..., 10, Jack, Queen, King. So there is a total of $13 \times 4 = 52$ cards. All the spades and clubs are black in colour. All the hearts and diamonds are red in colour. In real life, a pack of playing cards has a total of 54 cards: the two extra cards are Jokers.

Solution

(a) The total number of possible outcomes of this experiment is 52.

(b) (i) There are 26 black cards in the pack.

$$\text{P (drawing a black card)} = \frac{26}{52} = \frac{1}{2}$$

(ii) There are 2 red aces, i.e. the ace of hearts and the ace of diamonds.

$$\therefore \quad \text{P (drawing a red ace)} = \frac{2}{52} = \frac{1}{26}$$

(iii) There are 13 diamonds in the pack.

$$\therefore \quad \text{P (drawing a diamond)} = \frac{13}{52} = \frac{1}{4}$$

(iv) Since there are 13 diamonds in the pack, the remaining 39 cards in the pack are not diamonds.

$$\therefore \quad \text{P (drawing a card which is not a diamond)} = \frac{39}{52} = \frac{3}{4}$$

For Your Information

Casinos, lottery and operators in the gambling business make use of the other theory of probability to fix rules that will ensure that they are always on the winning side in the long run so that they will not go out of business.

Example 6

A circle is divided into four sectors coloured yellow, red, blue and green as shown in Fig. 12.4

A point is selected at random in the circle. Find the probability that it lies in the

(a) yellow sector,
(b) blue sector,
(c) black sector.

Is the probability of the point lying in the red sector or green sector greater? Why?

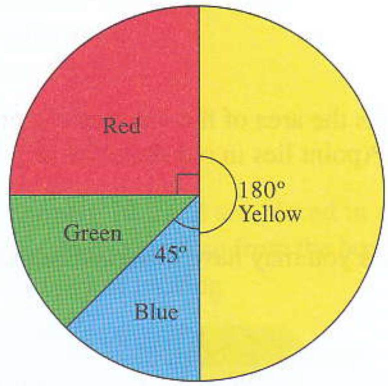


Fig. 12.4

Solution

Since a point is selected at random, any point in the circle will have the same chance of being selected. We assume the point will never fall on any of the lines separating the four sectors.

(a) Number of points in the yellow sector is proportional to the area of the sector which is also proportional to the angle of the sector.

$\therefore$ P(selecting a point in the yellow sector)

$$= \frac{\text{Area of the yellow sector}}{\text{Area of the circle}}$$

$$= \frac{\text{Angle of the yellow sector}}{360^\circ}$$

$$= \frac{180^\circ}{360^\circ} = \frac{1}{2}$$

(b) Similarly, P(selecting a point in the blue sector)

$$= \frac{\text{Area of the blue sector}}{\text{Area of the circle}}$$

$$= \frac{\text{Angle of the blue sector}}{360^\circ}$$

$$= \frac{45^\circ}{360^\circ} = \frac{1}{8}$$

Just for fun

Myrna, Alison and Shelly repeatedly take turns tossing a die. Myrna begins: Alison always follows Myrna; Shelly always follows Alison; and Myrna always follows Shelly. What is the probability that Shelly will be the first one to toss a six?

(c) Since there is no black sector in the circle,

$$\therefore \text{ P(selecting a point in the black sector)} = \frac{\text{Area of the black sector}}{\text{Area of the circle}}$$

$$= \frac{0}{\text{Area of the circle}} = 0$$

As the area of the red sector is greater than the area of the green sector, P(point lies in red sector) > P(point lies in green sector).

As you may have observed in the above examples,

> Probability is a value between 0 and 1 (inclusive).
>
> If the probability of an event occurring is 0, then it is an impossible event: it will never happen.
>
> If the probability of an event occurring is 1, then it is a certain event: it will definitely happen.

JOURNAL WRITING

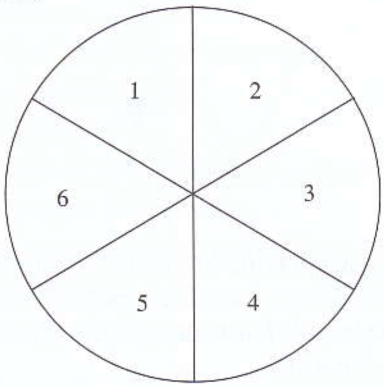
Probability theory was first used primarily in some problems on gambling. An Italian, Girolamo Cardano (1501 – 1576), wrote a gambler's manual which made use of probability theory. A French mathematician, Chevalier de Mere, posed a gambling problem to Blaise Pascal (1623 – 1662) in 1654. In response to this problem, Pascal and another French mathematician, Pierre Fermat (1601 — 1665), laid the foundations for the theory of probability.

This theory has become a powerful and widely applicable branch of mathematics. It has widespread use in business, science and industry. Its uses range from the determination of life insurance premiums to the description of the behaviour of molecules in a gas and also the prediction of outcomes in an election.

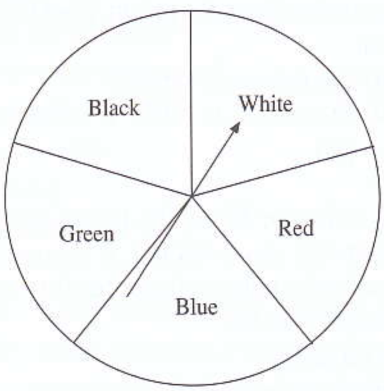
Search the Internet and find some other real life uses of probability theory. Write a journal based on your findings, and share them with the class.

Exercise 12a

1. For each of the experiments below, write down its sample space S and the total number of possible outcomes n(S).
   (a) Drawing a card from a box containing seven cards labelled A, B, C, D, E, F, G.
   (b) Drawing a disc from a bag containing 5 yellow discs and 2 green discs.
   (c) Selecting a point at random from the circular board shown in the diagram below.



   (d) Spinning the spinner shown in the diagram below.



   (e) Choosing a 3-digit number at random.
   (f) Tossing a standard 8-sided die.
   (g) Tossing a tetrahedral die with faces labelled 2, 3, 3 and 5 respectively.

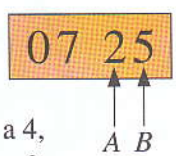
2. A box contains 8 balls numbered 1, 2, 3, 4, 5, 6, 7 and 8. A ball is selected at random from the box. Find the probability that the ball selected has
   (a) the number 8 on it,
   (b) an even number on it,
   (c) a prime number on it,
   (d) a multiple of 3 on it,
   (e) an integer on it,
   (f) a 2-digit number on it.

3. Cards with numbers 10 to 22 are placed in a box. A card is picked at random from the box. Find the probability of picking
   (a) an odd number,
   (b) a number between 12 to 20 exclusive,
   (c) a number bigger than 25,
   (d) a prime number less than 18,
   (e) a number smaller than 14,
   (f) a number divisible by 4.

4. (a) What is the probability of getting a '3' in a toss of a standard tetrahedral die?
   (b) What is the probability of getting a '7' in a toss of a 8-sided die with faces labelled 2, 3, 3, 4, 7, 7, 7, 9 respectively?

5. An IQ test consists of 80 MCQ. A question is chosen at random. Write down, giving your answers as a fraction in its lowest term, the probability that the question number chosen will
   (a) contain only a single digit,
   (b) be a perfect square,
   (c) be bigger than 67,
   (d) contain at least one figure 7,
   (e) be divisible by 5.

6. A man wakes up in the morning and notice that his digital watch reads 07 25.
   After lunch he looks at the watch again.
   What is the probability that



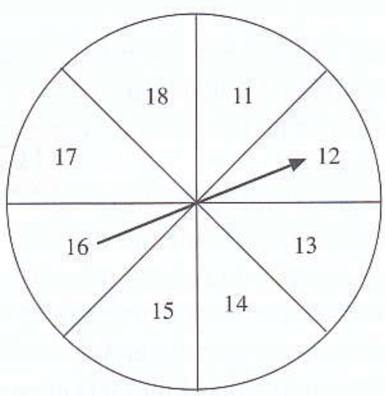
   (a) the number in column A is a 4,
   (b) the number in column B is a 8,
   (c) the number in column A is a 7,
   (d) the number in column A is less than 6,
   (e) the number in column B is greater than 5.

7. A two-digit number is written down at random. Find the probability that the number will be
   (a) smaller than 20,
   (b) even,
   (c) a multiple of 5.

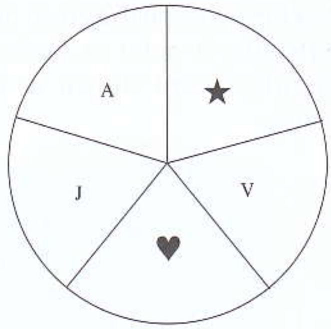
8. Each of the letters of the word MATHEMATICS is written on a card. All the eleven cards are well-shuffled and placed down on a table. If a card is turned over, what is the probability that the card bears
   (a) the letter 'M',
   (b) a vowel,
   (c) the letter 'P',
   (d) a consonant?

9. Each of the letters of the word TEACHER is written on a card. All the seven cards are well-shuffled and placed down on a table. If a card is turned over, what is the probability that the card bears
   (a) the letter 'E',
   (b) the letter 'S',
   (c) a vowel,
   (d) a consonant?

10. The diagram shows a spinner divided into 8 equal sectors. When the pointer is spun, what is the probability that the pointer will stop at
   (a) 16,
   (b) an even numbered sector,
   (c) a sector whose number is less than 14,
   (d) a sector whose number is 14 or less,
   (e) a sector whose number is divisible by 3,
   (f) a sector whose number is a single digit?



11. The diagram shows a spinner divided into 5 equal sectors. When the pointer is spun, what is the probability that the pointer will stop at
   (a) a sector whose label is ♥,
   (b) a sector whose label is a letter of the English alphabet,
   (c) a sector whose label is a vowel,
   (d) a sector whose label is a consonant?



12. A bag contains 40 marbles, 25 green ones and 15 red ones. A marble is picked at random from the bag. What is the probability of picking a red marble?

13. A group of 30 people consists of 9 men, 6 women, 12 boys and 3 girls. A person is chosen at random from the group. Find the probability that
   (a) the person is a male,
   (b) the person is either a woman, a boy or a girl.

14. In a class of 30 pupils, 12 are girls and two of them are short-sighted. If a pupil is selected randomly, what is the probability that the pupil chosen will be
   (a) a girl,
   (b) short-sighted?

15. Suppose a bag contains 20 sweets, of which 7 are toffee wrapped in green paper, 4 are barley sugar wrapped in red paper, 3 are toffee wrapped in red paper, and 6 are barley sugar wrapped in green paper. If a sweet is selected at random, calculate the probability that the sweet is

(a) a toffee,
(b) a barley sugar wrapped in red paper,
(c) wrapped in green paper.

16. A box of 2 dozen pencils contains 8 pencils with broken points. What is the probability of picking one pencil without a broken point?

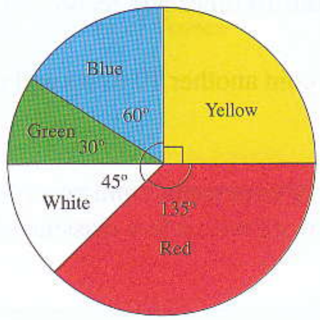
17. Peter has 5 Chinese books and 5 English books in his bag. Three of his ten books are science fiction which are in English. If a book is chosen at random,
find the probability of choosing
(a) an English book,
(b) a Malay book,
(c) either a Chinese book or an English book,
(d) a science fiction book,
(e) an English book which is not a science fiction book,
(f) a book which is not a science fiction book.

18. If we take a standard pack of 52 well-shuffled playing cards, what is the probability of drawing
(a) the ace of clubs,
(b) a king, queen or jack,
(c) a joker,
(d) a red card,
(e) a heart,
(f) a seven?

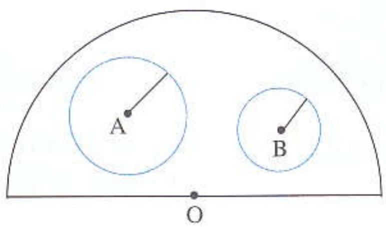
19. All the 26 red cards from a standard pack of playing cards are mixed thoroughly. A card is then drawn at random. Find the probability that the card drawn is
(a) the queen of hearts,
(b) the king of spades,
(c) either the king, queen or jack of diamonds,
(d) either the six of hearts or seven of diamonds,
(e) either a four of diamonds or the eight of clubs.

20. If we take a standard pack of 54 well-shuffled playing cards, what is the probability of drawing
(a) a joker,
(b) a black card,
(c) a club,
(d) a king or queen,
(e) the ace of spades,
(f) a two?

21. The diagram shows a circle divided into sectors of different colours. If a point is selected at random in the circle, calculate the probability that it lies in the
(a) yellow sector,
(b) red sector,
(c) blue sector,
(d) green or white sector.



✶ 22. The diagram shows a semicircle centre O and radius 10 cm, a circle centre at A with radius 4 cm and another circle centre B with radius 3 cm. A point is selected at random inside the semicircle. Find the probability that the point selected will lie inside or on



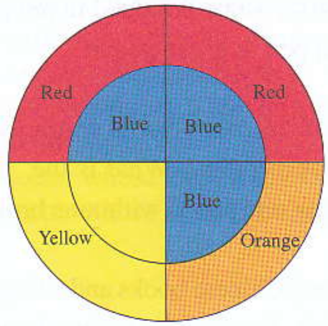
(a) the circle centre A,
(b) the circle centre B,
(c) neither circle A nor circle B.

∗ **23.** The diagram represents the target of a darts board. The target consists of two concentric circles of radius 3 cm and 6 cm. The target is coloured red, blue, yellow and orange as shown. A dartsman hits the target every time he throws a dart at the target and is equally likely to hit any part of the target. Find the probability that the dartsman will hit

(a) the blue region,

(b) the orange region,

(c) the region that is not yellow.



Explorations

1. Toss a coin 10 times and record the number of heads obtained in the table below.

2. Toss the coin another 10 times and record the total number of heads for all the 20 throws in the table below.

3. Repeat the experiment and complete the table below. If it is tedious to throw the coin 100 times, you can work with a few classmates with each of you throwing the coin 10 or 20 times.

Total Number of Tosses	10	20	30	40	50	60	70	80	90	100
Number of Head Tossed										
$y = \dfrac{\text{Number of Heads Tossed}}{\text{Total Number of Tosses}}$										

4. The probability of tossing a head is $\frac{1}{2}$.

Look at the ratio $\dfrac{\text{Number of Head Tossed}}{\text{Total Number of Tosses}}$ in the table above.

Do you get $\frac{1}{2}$ for each of the 10 ratios?

5. On a sheet of graph paper, taking 2 cm to represent 10 units on the x-axis and 2 cm to represent 0.1 unit on the y-axis, plot the ratio $\frac{\text{Number of Head Tossed}}{\text{Total Number of Tosses}}$ against the Total Number of Tosses. Then join the 10 points to form a line graph.

6. Draw the line $y = \frac{1}{2}$ on your graph to show the probability of tossing a head.

   (a) Do you notice the tendency of the points to get nearer to the line $y = \frac{1}{2}$ as the total number of tosses increases?

   (b) Do you expect the ratio $\frac{\text{Number of Head Tossed}}{\text{Total Number of Tosses}}$ to get closer to the probability of $\frac{1}{2}$ if you increase the total number of tosses from 1000 to 5000 to 10 000?

From the above exploration, we observe that the ratio $\frac{\text{Number of Head Tossed}}{\text{Total Number of Tosses}}$ will approach the probability of $\frac{1}{2}$ as the total number of tosses increases.

Example 7

Four cards bearing the numbers 2, 3, 4 and 5 are placed on the table. Two cards are selected from these four cards to form a two-digit number.
List the sample space. Find the probability that the number formed
(a) is divisible by 3,
(b) is greater than 33,
(c) is a multiple of 11,
(d) is less than 55.

Solution

The sample space is

$$S = \{23, 24, 25, 32, 34, 35, 42, 43, 45, 52, 53, 54\}$$

(a) The numbers which are divisible by 3 are 24, 42, 45 and 54.

$\therefore$ P(number is divisible by 3) $= \frac{4}{12} = \frac{1}{3}$

(b) The numbers which are greater than 33 are 34, 35, 42, 43, 45, 52, 53 and 54.

$\therefore$ P(number is greater than 33) $= \frac{8}{12} = \frac{2}{3}$

(c) There are no multiples of 11.
P(number is a multiple of 11) = 0

(d) All the 12 numbers are less than 55.

$\therefore$ P(number is less than 55) $= \frac{12}{12} = 1$

Exercise 12b

1. If you draw a card from a pack of 20 identical cards numbered 1, 2, 3, ..., 20, what is the probability of drawing
   (a) a prime number,
   (b) a non-prime number,
   (c) a composite number?

2. There are 3 identical red balls, 4 identical blue balls and 5 identical yellow balls in a box. What is the probability of drawing
   (a) a yellow ball,
   (b) a ball that is not yellow in colour,
   (c) a red ball,
   (d) a blue or a yellow ball?

3. The numbers 2, 3, 5 and 7 are written on 4 cards. Two of these cards are selected at random to form a two-digit number. List the sample space and hence find the probability that the number formed is
   (a) even,
   (b) divisible by 4,
   (c) prime,
   (d) greater than 55.

4. The numbers 5, 5, 6 and 6 are written on 4 cards. Two of these cards are selected at random to form a two-digit number. List the sample space and hence find the probability that the numbers formed is
   (a) divisible by 11,
   (b) a prime number,
   (c) greater than 55.

5. All the clubs are removed from a pack of ordinary playing cards.
   A card is drawn at random from the remaining cards in the pack. Find the probability of drawing
   (a) a red card,
   (b) a heart,
   (c) a picture card, i.e., jack, queen, or king,
   (d) a card that is not an ace.

6. The table shows the number of books given to 20 pupils.

2	3	0	2	1
3	0	1	3	4
1	2	4	2	1
4	3	2	2	3

   (a) A pupil is chosen at random. Find the probability that he or she receives 2 books.
   (b) A book is chosen at random. Find the probability that it is given to a student who receives 3 books.

7. A class has 16 boys and 24 girls. Of the 16 boys, 3 are left-handed and of the 24 girls, 2 are left-handed.
   (a) Mrs Tee, the form teacher, selects a pupil to run an errand. Assuming that she is equally likely to select a pupil, what is the probability that she selects
   (i) a boy,
   (ii) a left-handed pupil?
   * **(b)** While waiting for the pupil to clean the whiteboard, Mrs Tee selects one of the remaining pupils at random. Assuming that the first pupil she has sent away is a girl who is not left-handed, find the probability that she selects
   (i) a left-handed girl,
   (ii) a left-handed boy.

* 8. There are 23 boys and 35 girls in the school hall. After x boys and $x + 4$ girls left the hall, the probability of selecting a boy becomes $\frac{2}{5}$. Find the value of x.

* 9. There are 15 girls and x boys in the school parade square. One child is selected at random. If the probability that the child is a girl is $\frac{1}{5}$, calculate the value of x.

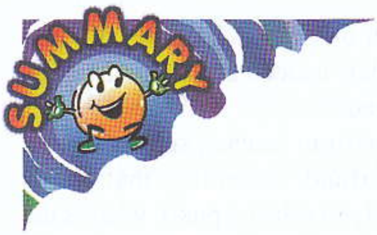
★**10.** There are 48 boys and 2 girls in the school field. After another x boys and $2x$ girls join the children in the school field, the probability of selecting a boy becomes $\frac{2}{5}$. Find the value of x.

★**11.** A bag contains 15 balls of which x are red.

(a) Write an expression for the probability that a ball drawn at random from the bag is red.

(b) When 5 more red balls are added to the bag, the probability of drawing a red ball becomes $\frac{3}{4}$. Find the value of x.

★**12.** There are 28 boys and 25 girls in the hall. After x girls left the hall, the probability of selecting a girl at random from those remaining in the hall becomes $\frac{3}{7}$. Calculate the value of x.



1. A **sample space** or **probability space** is the collection of all possible outcomes of a probability experiment.

2. An event E contains the outcomes from the sample space that favour the ocurrence of the event.

3. In a probability experiment with m equally likely outcomes, if k of these outcomes favour occurrence of an event E, then the **probability** of the event E happening is:

$$P(E) = \frac{\text{Number of favourable outcomes for event E}}{\text{Total number of possible outcomes}} = \frac{n(E)}{n(S)} = \frac{k}{m}$$

   where $n(E)$ is the number of favourable outcomes in the event E and $n(S)$ is the total number of possible outcomes.

4. For any event E, $0 \leq P(E) \leq 1$.
   $P(E) = 0$ if and only if the event E cannot possibly occur.
   $P(E) = 1$ if and only if the event E will certainly occur.

Review Examples 12

Example 1

The table below shows the number of stamps of the same size in an envelope.

Value of stamp in cents	*23*	*30*	*40*	*50*
Number of stamps	*10*	*20*	*32*	*28*

A stamp is picked at random from the 90 stamps in the envelope.
Find the probability that the value of the selected stamp is
(a) more than 30 cents,
(b) 30 cents or less,
(c) less than 20 cents,
(d) more than 22 cents.

Solution

(a) There are (32 + 28) stamps, each of which has a value of more than 30 cents.

$\therefore$ P(value of stamp is more than 30 cents) $= \frac{60}{90} = \frac{2}{3}$

(b) P(value of stamp is 30 cents or less) = 1 – P(value of stamp is more than 30 cents)

$$= 1 - \frac{2}{3}$$

$$= \frac{1}{3}$$

(c) None of the stamps has a value less than 20 cents.
$\therefore$ P(value of stamp is less than 20 cents) = 0

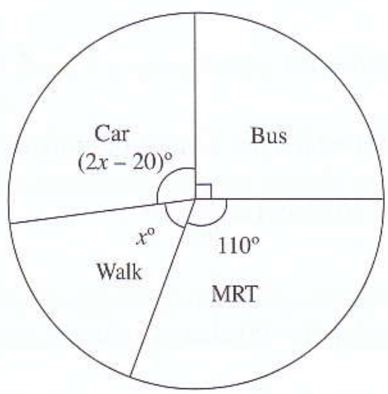
(d) Each of the 90 stamps has a value of more than 22 cents.
$\therefore$ P(value of stamp is more than 22 cents) = 1

Review Questions 12

1. A fair die is tossed once. Find the probability of obtaining
   (a) an even number,
   (b) a prime number,
   (c) a number that is divisible by 4.

2. A wheel of a slot machine has 4 slots of cherries, 7 slots of oranges, 9 slots of peaches and 2 slots of grapes. Assuming that the wheel is fair, find the probability that the wheel will stop at the slot with
   (a) an orange,
   (b) a peach,
   (c) a pineapple,
   (d) either a grape or a cherry.

3. The pie chart below shows the proportion of pupils travelling to school using the various types of transport.
   (a) Find the value of x.
   (b) If a pupil is selected, find the probability that this pupil travels to school by
      (i) bus,
      (ii) car,
      (iii) helicopter.



4. A research paper found that 480 out of 680 lung cancer patients were cigarette smokers. If a patient is selected at random from these 680 lung cancer patients, find the probability that the patient is not a cigarette smoker.

5. In a class of 14 girls and 26 boys, a pupil is selected at random.
   (a) Find the probability that the pupil selected is a girl.
   ∗ (b) If the girl selected leaves the class and a second pupil is then selected at random, find the probability that the pupil selected is a boy.

6. A bag contains 7 identical red marbles and 11 identical white marbles.
   (a) A marble is drawn at random from the bag. Find the probability that the marble is
      (i) red,
      (ii) green,
      (iii) either white or red.
   ∗(b) 12 yellow marbles are added to the bag of 18 marbles in (a) and these are thoroughly mixed. A new marble is then drawn from the bag. Calculate the probability that the new marble drawn is
      (i) red,
      (ii) green,
      (iii) yellow.

7. Each letter of the word "PROBABILITY" is written on a card and these cards are thoroughly mixed in a bag. A card is picked randomly from the bag.
   (a) Find the probability that the card picked contains
      (i) the letter B,
      (ii) a vowel,
      (iii) the letter K.
   ∗(b) A card with the letter I is removed from the bag. Find the probability of picking a card with
      (i) the letter I,
      (ii) the letter B,
      (iii) a consonant.

8. The numbers 2, 5 and 7 are written on three cards. One or more of these cards are selected to form a number that consists of either a one-, two- or three-digit number.
   List the sample space of this experiment. Hence, find the probability that the number formed
   (a) consists of two digits,
   (b) is greater than 55,
   (c) is divisible by 6.

9. The following table shows the amount of pocket money received by 40 students in a class in a day.

Amount in $	1.50	2.00	2.50	3.00	3.50	4.00
No. of students	4	8	13	7	5	3

   A student is selected from the class. Find the probability that the student received
   (a) $2.00,
   (b) more than $3.00,
   (c) less than or equal to $2.50,
   (d) not more than $2.00.

10. The number of children in a family in a new block of apartments is shown in the table below.

No. of children	0	1	2	3	4	5
No. of families	13	29	37	22	8	1

A family is selected from the apartment block. Calculate the probability that the family selected has
(a) more than 2 children,
(b) less than 2 children,
(c) not more than 3 children,
(d) not less than 4 children.

11. The number of rotten oranges found in 50 boxes is as shown below.

No. of rotten oranges	0	1	2	3	4
No. of boxes	18	15	8	6	3

A box is selected at random. Calculate the probability that the box contains
(a) 3 rotten oranges,
(b) at least 2 rotten oranges,
(c) more than 1 rotten orange,
(d) no less than 4 rotten oranges,
(e) more than 4 rotten oranges.

12. The table below shows the average monthly amount Singaporean gamblers spent on gambling.

Average amount $\$x$	$x \leq 20$	$20 < x \leq 50$	$50 < x \leq 100$	$100 < x \leq 200$	$x > 200$
Percentage of people	40	28	20	7	5

If a gambler is selected at random, what is the probability that he spends
(a) no more than $20 per month,
(b) more than $100 per month,
(c) between $20 and $200 per month?

13. The table below shows the mode of HIV transmission of 242 HIV-positive patients in 2003.

Mode of transmission	Heterosexual	Homosexual	Bisexual	Intravenous	Others
No. of HIV patients	183	40	14	4	1

If a HIV-positive patient is selected from these 242 patients, what is the probability that the patient contracted the disease through

(a) heterosexual means,

(b) homosexual means,

(c) intravenous means?

14. The table below shows the percentage of households in the lowest 20% income group with specified consumer durables in 1998 and 2008.

Types of consumer durables	PC	Internet	Cellphone	Air-Con	Fridge	CD/VCD Player
Percentage of households in 1998	17	47	21	30	95	32
Percentage of households in 2008	31	20	63	42	96	61

(a) If a household is selected from this income group in 1998, what is the probability that the household has

(i) internet access,

(ii) air-conditioners?

(b) If a household is selected in 2008 from this income group, what is the probability that this household

(i) possesses a PC,

(ii) owns a CD/VCD player?

15. A girl types 3 letters and 3 different addresses on 3 envelopes. She puts the letters into the envelopes randomly and sends them to 3 of her friends, Alice, Bettey and Carol.
What is the probability that

(a) only one of her friends receive the correct letter,

Revision Exercise IV No. 1

1. If $\frac{c}{y} = k\left(1 + \frac{x}{y}\right)$, express k in terms of c, x and y. Find k where $c = 5$, $x = 1\frac{1}{2}$ and $y = 2\frac{1}{2}$.

2. A car travels 360 km in 5 hours. What is the speed of the car in
   (a) kilometres per hour,
   (b) metres per second?

3. Factorise the following expressions:
   (a) $x^2 - 21x + 20$
   (b) $x^2 + 13x + 36$
   (c) $1 + 3x - 5k - 15kx$

4. **(a)** If all birthdays are equally likely to occur in any month of any year, find the probability that two pupils selected at random from your class do not have birthdays occurring in the same month.
   (b) If all birthdays are equally likely to occur in any day of any year, find the probability that two pupils selected at random from your school will have birthday occuring in the same day of the year. (Take 1 year = 365 days)

5. Given that $\mathcal{E} = A \cup B$, $n(A \cup B) = 48$, $n(A \cap B) = 9$ and $n(B) = 31$, use a Venn diagram to illustrate the above information and hence find the value of $n(A)$.

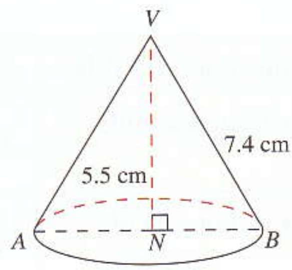
6. Simplify
   (a) $\frac{(2x-1)}{3} - \frac{5(x-3)}{6} - \frac{3}{4}$,
   (b) $3[(a - 2b) - (2a - b)] - 2[(2a + b) - (a + 2b)]$,
   (c) $\frac{1}{x} + \frac{2}{x^2} + \frac{3}{x^3}$,
   (d) $\frac{a-b}{a} - \frac{b-a}{b}$.

7. **(a)** A pyramid stands on a square base of sides 5.8 cm. If the height of the pyramid is 10 cm, find its volume. Give your answer correct to 3 significant figures.
   (b) Ten dozen lead spheres, each of diameter 3 cm, are melted and recast into a cylinder 12 cm in diameter. Calculate the height of the cylinder. (Take $\pi = 3.142$.)
   (c) The volume of a cone is 56.8 cm^3 and its base radius is 14.7 cm. Find its height.

8. The following are marks scored by 9 pupils in a Science test marked out of a total of 15:

   4, 9, 11, 8, 10, 5, 11, 14, 7

   Find **(a)** the mode,
   (b) the median,
   (c) the mean of this set of marks.

9. 

   The figure shows a cone of height 5.5 cm and a slant height of 7.4 cm.
   (a) Calculate
   (i) the diameter of the base of the cone,
   (ii) the volume of the cone, giving your answer correct to 1 decimal place.
   (b) If the cone is made of solid material of density 3.4 g/cm^3, calculate the mass of the cone. Give your answer correct to 1 decimal place.

10. Copy and complete the following table of values for $y = x^2 - 4x - 2$.

x	−1	0	1	2	3	4	5	6
y		−2		−6		−2		10

Using a scale of 2 cm to represent 1 unit on the x-axis and 2 cm to represent 2 units on the y-axis, draw the graph of $y = x^2 - 4x - 2$.

Use your graph to estimate
(a) the value of y when $x = 3.5$,
(b) the minimum value of $x^2 - 4x - 2$.

Revision Exercise IV No. 2

1. Simplify **(a)** $\frac{x}{y} + \frac{z^2}{xy} + \frac{y}{x}$,

(b) $\frac{3x-2y}{2} - \frac{x+y}{3} - 1$.

2. Solve the following equations:

(a) $\frac{x+2}{6} + \frac{x-2}{2} = \frac{4-4x}{12}$

(b) $\frac{4}{5x} = \frac{1}{10} + \frac{7}{10x}$

(c) $\frac{3}{x+3} = \frac{4}{x+9}$

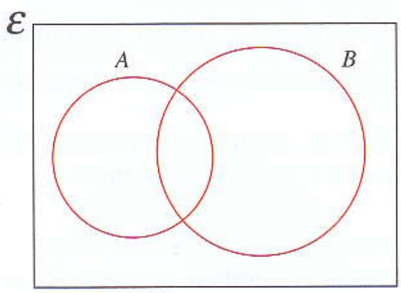
(d) $\frac{15}{x-8} = \frac{6}{20-x}$

3. On three copies of the Venn diagram below, shade:

(a) $A \cap B$

(b) $A \cap B'$

(c) $A' \cup B$



4. A two-digit number is written down on a piece of paper. Find the probability that the number written is

(a) an odd number,

(b) divisible by 4,

(c) greater than 88,

(d) less than 18.

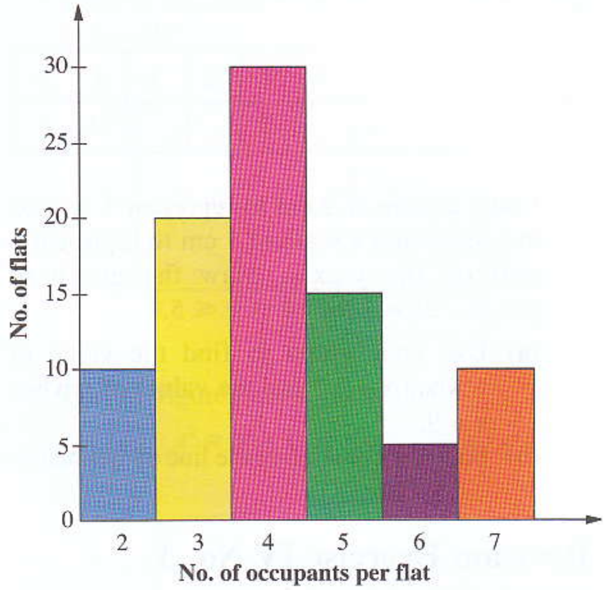
5. Expand the following:

(a) $(2x + y)(3x - 5y)$

(b) $(2x - 3y)(2x + y - 4)$

6. Solve the equation $x^3 - 5x^2 + 6x = 0$.

7.



The bar chart shows the number of occupants per HDB flat in a particular location.

(a) State the modal number of occupants per flat.

(b) Calculate the mean number of occupants per flat.

(c) If this information is to be represented by a pie chart, find the angle of the sector representing flats with 7 occupants.

8. Simplify the following:

(a) $\frac{1}{x} + \frac{1}{y}$

(b) $\frac{1}{x+2} - \frac{1}{x+3}$

(c) $\frac{2}{x+y} + \frac{3}{x-y}$

9. The mean of five numbers is 34. Three of the numbers are 29, 26 and 35. If the remaining two numbers are in the ratio 1 : 3, find the numbers.

10. Copy and complete the table of values for $y = x^2 - 2x + 3$.

x	−3	−2	−1	0	1	2	3	4	5
y	18		6		2			11	

Using a scale of 2 cm to represent 1 unit on the x-axis and a scale of 1 cm to represent 1 unit on the y-axis, draw the graph of $y = x^2 - 2x + 3$ for $-3 \leqslant x \leqslant 5$.

(a) Use your graph to find the value of y when $x = 2.5$ and the values of x when $y = 9$.

(b) State the equation of the line of symmetry.

Revision Exercise IV No. 3

1. If \$6560 is divided into three shares in the ratio $1\frac{1}{2} : 2\frac{1}{3} : 3$, how much is the smallest share?

2. (a) If $\frac{a-b}{c-b} = \frac{x}{y}$, make b the subject of the formula. Find the value of b when $a = -1$, $c = 2$, $x = 10$ and $y = 6$.

(b) Solve the equation $(2x + 5)(2x - 5) = (x - 1)(4x + 3)$.

3. If 5 kg of tea and 2 kg of coffee cost \$81 and 1 kg of tea and 4 kg of coffee cost \$54, find the price of 1 kg of tea and 1 kg of coffee.

*4. A bag contains 48 marbles of which x are white and the rest are red. When 8 more white marbles are added to the bag, the probability of drawing a white marble from the bag becomes $\frac{3}{7}$. Find

(a) the value of x,

(b) the number of red marbles in the bag.

5. If $\mathcal{E}$ = {quadrilaterrals}, P = {parallelograms}, Q = {squares}, R = {rectangles}, simplify

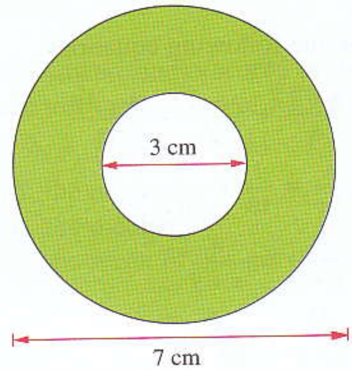
(a) $P \cap R$ (b) $P \cap Q$

(c) $P \cup R$ (d) $Q \cap R$

6. Find the mean and the median age of the following nine children:

12 yr 9 mth, 11 yr 4 mth, 14 yr 3 mth,
15 yr 7 mth, 12 yr 2 mth, 13 yr 8 mth,
10 yr 6 mth, 16 yr 11 mth, 13 yr 7 mth.

7.



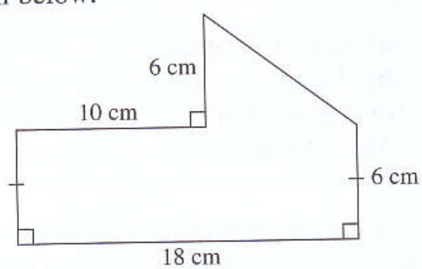
The diagram shows a cylindrical washer of outer diameter 7 cm and inner diameter 3 cm.

(a) Taking π to be 3.14, calculate the area of the shaded region.

(b) Given that the washer has a thickness of $\frac{1}{4}$ cm and is made of a metal of density 4 g/cm^3, calculate the mass of the washer correct to the nearest gram.

8. The mean of 12 numbers is 8 and the mean of the first 5 numbers is 7.6. Find the mean of the last 7 numbers.

9. The cross-section of a solid block of wood is shown below.



(a) If the length of the piece of wood is 1 m, find its volume.

(b) If 1 cm^3 of wood weighs 0.7 g, find the weight of the block of wood in grams.

10. Copy and complete the following table of values for $y = x^2 - x - 5$.

x	-2	-1	0	1	2	3
y	1		-5	-5		

Plot the graph of $y = x^2 - x - 5$ using a scale of 2 cm to represent 1 unit for both the x- and y-axes.

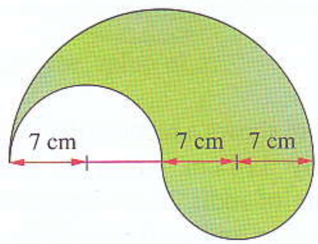
Use your graph to find

(a) the value of x for which y has the least value,
(b) the value of y when $x = 1.4$,
(c) the values of x when $y = 0$.

Revision Exercise IV No. 4

1. The lengths of the two diagonals of a rhombus are 16 cm and 12 cm respectively. Find the length of one side of the rhombus.

2. The diagram consists of three semicircles whose radii are 7 cm, 7 cm and 14 cm.



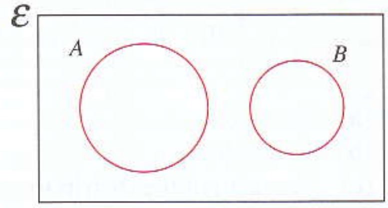
Calculate, taking $\pi = 3.142$,

(a) the circumference of the shaded region,
(b) the area of the shaded region.

3. 50 discs, numbered from 20 to 69, are placed in a box and one disc is drawn out at random. Find the probability that the number on the disc

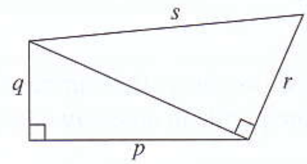
(a) is greater than 54,
(b) is less than 33,
(c) includes the digit 8.

4. Given the Venn diagram below and that $n(\mathcal{E}) = 35$, $n(A) = 20$ and $n(B) = 12$, find $n(A \cup B)'$.

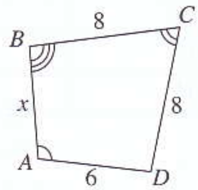


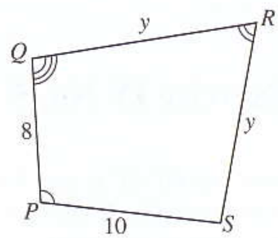
5. For the figure below, show that

$$s^2 = p^2 + q^2 + r^2.$$



6. In the diagram, $ABCD$ is similar to $PQRS$. Calculate the values of x and y.





7. Three of the interior angles of a pentagon are 96°, 110° and 126° while the other two angles are each equal to $2x°$. Find the value of x.

8. The distribution of the length (x cm) of 80 leaves is given in the table below:

x (in cm)	10	11	12	13	14	15	16
Frequency	6	10	20	22	14	6	2

Find **(a)** the mode,
(b) the median,
(c) the mean of the distribution, giving your answer correct to 1 decimal place.

9. In $\triangle ABC$, $\hat{C} = 90°$, $AB = 13$ cm and $AC = 5$ cm. Find the length of BC and the area of $\triangle ABC$.

10. The table below shows the number of mistakes made by the pupils in an essay competition.

No. of Mistakes	No. of Pupils
1 – 3	12
4 – 6	18
7 – 9	25
10 – 12	13
13 – 15	7
16 – 18	4

Calculate the mean number of mistakes made, giving your answer correct to 1 decimal place.

Revision Exercise IV No. 5

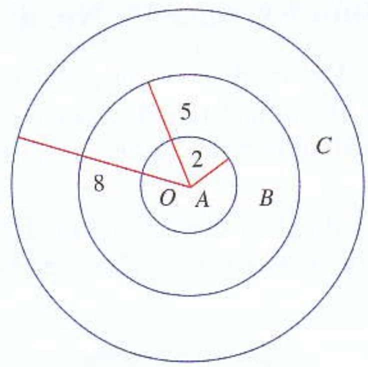
1. The average speed of a car for a 425-km journey was 50 km/h. If the car stopped for 90 minutes on the road, find the average speed of the car while travelling.

2. The commission received by a salesman is $7\frac{1}{2}\%$ on all his sales. If his sales totalled \$2240, find his commission.

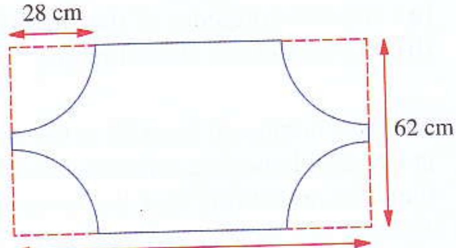
3. Given that $s = ut + \frac{1}{2}at^2$, express a in terms of s, u and t. Find the value of a when $s = 10\frac{1}{2}$, $u = 1\frac{1}{2}$ and $t = 3$.

4. A man paid \$51 for 3 shirts and 12 pens. If a shirt costs \$4.50 more than a pen, find the price of each shirt and each pen.

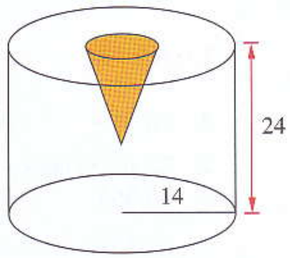
5. The diagram shows a dart board with centre O. The radius of the innermost circle is 2 cm, the radius of the middle circle is 5 cm while the outermost circle has a radius of 8 cm. If a dart thrown is likely to land at a point on the board, find the probability that the dart will land in region
(a) A
(b) B
(c) C



6. The figure shows a rectangular piece of wood 62 cm by 118 cm. From its four corners, quadrants of radii 28 cm have been cut off. Find the area and perimeter of the remaining piece of wood. (Take $\pi = 3.142$.)



7. From a solid cylinder whose height is 24 cm and radius 14 cm, a conical cavity of height 12 cm and base radius 10 cm is hollowed out. Find the volume of the remaining solid, giving your answer correct to 4 significant figures.



8. The following is a set of marks scored by 12 students in a Mathematics test marked out of a total of 20.

18, 4, 7, 11, 15, 8, 14, 11, 18, 19, 11, 20

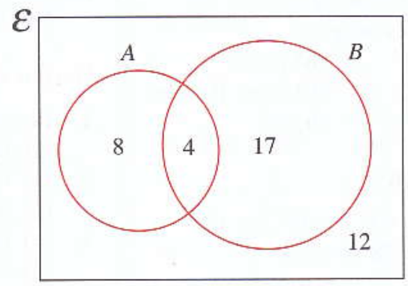
(a) Find the mode.
(b) Find the value of the median.
(c) Calculate the mean mark of the 12 students.
(d) Another boy, who was absent on that day, was given the same test the following day and he scored x marks. With this new mark, the mean now becomes $13\frac{2}{13}$. Find the value of x.

9. The lifespan (days) of a certain kind of insect is given in the following table.

Lifespan (days)	Frequency
1 – 4	32
5 – 8	26
9 – 12	18
13 – 16	11
17 – 20	5

Calculate the mean lifespan of the insects. State the modal class.

10. The Venn diagram below shows the number of elements in sets A and B.



Find:
(a) $n(A)$
(b) $n(A' \cap B')$
(c) $n[(A \cap B)']$

Answers

Exercise 1a

1. (a) and (f); (b) and (j); (c) and (e); (d) and (g); (i) and (k)
2. (a) 5 (b) DC (c) AD, 2 (d) BC, 5 (e) ABC, 90
3. (a) 3.5 (b) VZ (c) WX, 3.5 (d) ZY, 2 (e) YX, 2 (f) VWX, 90
4. $\hat{A} = 100°$, $\hat{C} = 75°$
   $AB = 3.5$ cm, $CD = 2.4$ cm
   $\hat{A} = 65°$, O = 120°
   $MN = 5$ cm, LO = 3 cm
5. (a) $\triangle ABC \equiv \triangle PQR$
   (b) $\triangle XYZ \equiv \triangle FEG$
   (c) Not congruent
6. (a) 56° (b) 16 cm
7. (a) 28° (b) 9 cm. Parallel
8. (a) 6 cm (b) 96°

Exercise 1b

1. (a) $x = 90°, y = 35°, z = 55°$
   (b) $x = 28°, y = 34°$
   (c) $x = 7.2, y = 10.8$
   (d) $x = 9.6, y = 5\frac{5}{6}$
2. (a) $x = 9, y = 85°$
   (b) $x = 95°, y = 52°, z = 4.8$
   (c) $x = 80°, y = 10.5$
3. $x = 16, z = 1\frac{7}{8}$
4. $x = 270, y = 100, z = 100°$

Exercise 1c

1. 9 cm, 4 cm
2. 7.5 cm, 10 cm, 3.5 cm
3. (a) 2 (b) 8 cm, 10 cm
4. 3 cm, 7 cm
5. (a) 4.5 m by 3.75 m; 16.875 m^2
   (b) 3 m by 2.25 m; 6.75 m^2
   (c) 78.75 m^2
6. (a) 36 m (b) 4 cm
7. 1 cm : 2 m; 12.5 cm
8. 21 cm
9. 58.3 m
10. (a) 80.2 m (b) 68.5 m
11. (a) 5 km (b) 1.5 cm
12. (a) 5 cm (b) 7.6 cm (c) 2.5 cm (d) 0.5 cm
13. (a) (i) 1 km (ii) 3.75 km (iii) 300 m (iv) 13 km
    (b) (i) 8 cm (ii) 30 cm (iii) 0.5 cm (iv) 3.2 cm
14. 1.1 km
15. (a) 12 km^2 (b) 18 km^2
16. 3400 ha
17. 2.5 cm^2
18. (a) 1 cm^2 (b) 2 cm^2 (c) 5 cm^2 (d) 25 cm^2
19. 3100 cm^2
20. (a) 2 km (b) 56 cm (c) 3 km^2
21. 126 km^2
22. (a) 1.2 km, 0.8 km
    (b) 0.96 km^2

Review Questions 1

1. (a) and (h); (b) and (f); (c) and (g); (d) and (e)
2. (a) 70° (b) 60° (c) 50° (d) 8 cm
3. (a) 100° (b) 70° (c) 95° (d) 95° (e) 6 cm
4. (a) 10.9 cm (b) 75°
5. (a) 4.2 cm (b) 92°
6. (a) 6 cm (b) 8 cm (c) 53°
7. (a) 50° (b) 68° (c) 62°
8. (a) 60° (b) 7.5 cm
9. 48 cm
10. 4.5 m
11. (i) Yes
    (ii) No
    (iii) 45
12. (a) No. Not all ratios of corresponding sides equal.
    (b) 7
13. (a) 20% (b) 1.2 m
14. (a) 20 m (b) 50 cm
15. (a) 40 m (b) 60 cm
16. (a) 24.3 km (b) 1.54 cm^2
17. (a) 1 cm : 250 m
    (b) 750 m (c) 32 cm
18. (a) 10.5 km (b) 1 : 150 000
    (c) 36 cm^2
19. (a) 6 km (b) 14 cm
    (c) 1 km^2
20. (a) 11.4 km (b) 7 cm
    (c) 2160 ha
21. (a) 98 m (b) 25 cm^2
    (c) 4 ha
22. (a) 312 km
    (b) \$82.80
    (c) 2 h 42 min
    (d) 42 km/h
    (e) \$17.50

Exercise 2a

1. (a) $41\frac{2}{3}$ (b) 72 books
2. 7.5 m; 32 books
3. (a) \$64 (b) \$60
   (c) \$ $\frac{ac}{b}$
4. $3\frac{3}{5}$ kg
5. (a) $2\frac{6}{7}$ (b) $3\frac{8}{9}$
   (c) 33 (d) 18
   (e) 8 (f) $\sqrt{8}$
6. (a) 7 : 5 (b) 3 : 8
   (c) 9 : 5 (d) 55 : 24
7. 8 kg
8. 1.5 m

Exercise 2b

1. (a) $y = 3x$ (b) 30
   (c) 3
2. (a) $x = 1.5y$ (b) 10.5
   (c) 8
3. (a) $Q = 7P$ (b) 35
   (c) 6
4. 17.5
5. 4.5
6. 4
7. (a) $y = 12, 21; x = 8, 11.5$
   (b) $y = 1, 5; x = 36, 44$
   (c) $y = 2.4, 6.6; x = 8, 9.5$
8. (a) $C = \frac{5}{3}d$ (b) \$75
   (c) 72 km
9. (a) 137.2 Newtons
   (b) 22 kg
10. (a) 60 (b) 4.8
11. (a) 22.5 (b) 10
12. (a) $y = 4x$
13. (a) $z = 8y$
14. (a) \$13 200
    (b) 380
    (c) $C = 41n + 5000$
    (d) No

15. (a) \$1360
(b) 135
(c) $D = 8n + 600$
(d) No
16. 95 tonnes

Exercise 2c

1. (a) y and x^2
(b) y and $\sqrt{x}$
(c) y^2 and x^3
(d) q and p^2
(e) q and $p\sqrt{p}$
(f) m^3 and n^2
(g) n and $(m-1)^2$
(h) $y - 2$ and $(x + 1)^3$
(i) $y - 1$ and x
(j) $2y + x$
(k) y and $\frac{1}{x}$
(l) y and $\frac{1}{x}$
2. (a) $y = 2x^2$
(b) 50
(c) ± 4
3. (a) $x = 4y^3$
(b) 864
(c) 3
4. (a) $z^2 = 2w$
(b) ± 6
(c) 12.5
5. 63
6. 27
7. 5
8. $y = 36, 225; x = 7, 2.5$
9. $v = 81, 192; r = 5, 7$
10. $m = 0.016, 0.686; r = 0.5, 1.8$
11. (a) $l = 24.5\ T^2$
(b) 612.5 cm
(c) 2 s
12. 5 cm, 36 hours

Exercise 2d

1. (b), (c), (e)
2. 40 min
3. 5 books
4. 20 days
5. $17\frac{1}{2}$ h
6. (a) 840 cattle
(b) 40 days
7. 28
8. 2 days
9. 10.5 hours
10. 36 hours
11. 21 min
12. 11

Exercise 2e

1. (a) $y = \frac{10}{x}$ (b) 1.25
(c) 1
2. (a) $x = \frac{200}{y}$ (b) 8
(c) $\frac{1}{2}$
3. (a) $z = \frac{0.5}{x}$ or $z = \frac{1}{2x}$
(b) 0.1
(c) 2.5
4. 6
5. $\frac{1}{2}$ or 0.5
6. 1.4
7. (a) $y = 8, 1\frac{1}{3}; x = 5, 1$
(b) $y = 6, 4.8; x = 0.5, 8$
(c) $y = 12, 1.44; x = 4.5, 14.4$
8. (a) $f = 300\,000\,000w$
(b) 600 kHz
(c) 375 m
9. (a) 2 amperes
(b) 2 ohms
10. (a) 200
(b) 8
11. (a) 32
(b) 4 hours
12. $y = \frac{2}{x}$
13. $z = \frac{6}{x}$

Exercise 2f

1. (a) y and x^2
(b) y and x^3
(c) y and $\sqrt{x}$
(d) q^2 and p^3
(e) n and $m\sqrt{m}$
(f) z and $w-1$
(g) y and $(x+1)^2$
(h) $y+3$ and x^2
(i) $y+1$ and x^2
(j) $2y^3 - 4$ and x
(k) y and $\frac{1}{x}$
(l) y^2 and $\frac{1}{\sqrt{x}}$
2. (a) $y = \frac{32}{x^2}$ (b) $\frac{8}{9}$
(c) ± 2
3. (a) $P = \frac{400}{Q^3}$ (b) 6.25
(c) 5
4. (a) $z = \frac{27}{\sqrt{x}}$ (b) 6.75
(c) 81
5. 2.5
6. $s = 4, 2; t = 36, \frac{1}{4}$
7. $n = 3; y = 1.25; x = 20$
8. (a) $F = \frac{40}{d^2}$ (b) 1.6
(c) 1.26
9. 20 cm

Review Questions 2

1. (a) $y = 3x$ (b) 4
2. 28; $6\frac{1}{14}$
3. $\frac{2}{3}$; $3\frac{3}{4}$
4. (a) $a + 300b = 29$
$a + 700b = 57$
(b) $a = 8, b = 0.07$
(c) \$30.40
5 $d = 3t^2$
6. $V = 4x^3$; 864; 10
7. $13\frac{1}{2}$
8. 5; 8
9. (a) 28 (b) ± 10
10. 81
11. \$225
12. (a) $y = \frac{12}{x}$ (b) 2
13. (a) $z = \frac{18}{\sqrt{x}}$ (b) 3.6
14. $\frac{3}{4}$
15. 16
16. \$45, \$80
17. $s = 5t^2$
18. $\frac{1}{25}$
19. 7

Exercise 3a

1. (a) $6a + 3b$
   (b) $12x + 16y$
   (c) $15c - 20d$
   (d) $35h - 14k$
   (e) $-6h - 21k$
   (f) $-24a - 40b$
   (g) $-42x + 18y$
   (h) $-27h + 18k$
   (i) $-6 - 8x$
   (j) $-10 + 15x$
2. (a) $10x^2 + 2xy$
   (b) $6a^2 + 21ab$
   (c) $18x^2 - 6xy$
   (d) $5a^2 - 5ab$
   (e) $-5xh - 3xk$
   (f) $-6a^2 - 9ab$
   (g) $-4h^2 + 6hk$
   (h) $-8m^2 + 20mn$
   (i) $14a^2 - 21ab$
   (j) $6hk + 21k^2$
3. (a) $5x + 17$ (b) $13x + 27$
   (c) $3h + 7$ (d) $10x - 5$
   (e) $29h - 42$ (f) $36x - 28$
   (g) $3 - 4h$ (h) $30x - 32$
   (i) $63 - 33x$ (j) $13x - 16$
4. (a) $3x^2 + 5x$ (b) $5x^2 + 7x$
   (c) $9x^2 + 4x$ (d) $29x - 3x^2$
   (e) $-13x^2 - x$ (f) $7p^2 - 7pq$
   (g) $-3a^2$ (h) $10ab - 8a^2$
   (i) $7p^2 - 12pq$ (j) $6x^2 + 19xy$

Exercise 3b

1. (a) $a^2 + 10a + 21$
   (b) $a^2 - 11a + 30$
   (c) $c^2 + 5c - 14$
   (d) $x^2 - 8x - 9$
   (e) $n^2 + 18n + 81$
   (f) $n^2 - 18n + 81$
   (g) $x^2 + 2xy - 3y^2$
   (h) $b^2 - 16c^2$
   (i) $m^2 - mp - 6p^2$
   (j) $e^2 + 6ef + 5f^2$
2. (a) $x^2 + 4x + 8$
   (b) $2x^2 + 16x - 7$
   (c) $8x^2 + x - 3$
   (d) $-5x^2 + 28x - 24$
   (e) $-2x^2 + 5x + 4$
   (f) $-x^2 + 11x - 13$
   (g) $x^2 + x - 8$
   (h) $3x^2 - 15x + 12$
   (i) $5x^2 - 10x + 3$
   (j) $-3x^2 + 29x - 5$
3. (a) $x^3 + 3x^2 + 3x + 2$
   (b) $x^3 - 2x - 1$
   (c) $x^3 + x^2 - 3x + 1$
   (d) $x^3 - 4x^2 + 7x - 6$
   (e) $a^3 + 6a^2 + 7a - 6$
   (f) $a^3 - 6a^2 + 13a - 12$
   (g) $x^3 - x^2 - 13x + 28$
   (h) $x^3 - x^2 - 21x + 5$
   (i) $x^3 - 4x^2 - 23x - 6$
   (j) $2x^3 + 7x^2 + x - 1$

Exercise 3c

1. (a) $a^2 + 8a + 16$
   (b) $m^2 + 14m + 49$
   (c) $x^2 - 18x + 81$
   (d) $4x^2 - 4x + 1$
   (e) $1 - 6x + 9x^2$
   (f) $4 + 12k + 9k^2$
   (g) $p^2 - 4pq + 4q^2$
   (h) $c^2 - 8cd + 16d^2$
   (i) $25x^2 - 30x + 9$
   (j) $16 - 24t + 9t^2$
2. (a) $4x^2 - 25$
   (b) $9x^2 - 16$
   (c) $25x^2 - 4$
   (d) $4x^2 - 9$
   (e) $49 - 4x^2$
   (f) $4h^2 - k^2$
   (g) $36x^2 - 25$
   (h) $25p^2 - q^2$
   (i) $4x^2 - 121$
   (j) $x^4 - y^2$
3. (a) 249 996 (b) 89 975
   (c) 9996 (d) 1 447 209
   (e) 811 801 (f) 808 201
   (g) 795 664 (h) 648 025
   (i) 4900 (j) 6400
4. 24
5. 4
6. 118
7. 601
8. (a) 59
   (b) 328
   (c) 88 888 888

Exercise 3d

1. (a) $2(4x + 1)$
   (b) $4(x - 5)$
   (c) $2x(x + 3)$
   (d) $5y(x - 5z)$
   (e) $3x^2(x - 4y)$
   (f) $2x(5x - 9)$
   (g) $4xy(1 - 3z)$
   (h) $7y(2x - 7y)$
   (i) $3ab(a + 2b + c^2)$
   (j) $5x(2x^2 + 3y + 4z)$
2. (a) $(1 + y)(x + 2y)$
   (b) $(x + 2)(x + 3y)$
   (c) $(p + q)(2s + t)$
   (d) $(p + q)(3s + 2t)$
   (e) $(x + y)(z + y)$
   (f) $(x + 2y)(x - 3)$
   (g) $(2x + z)(3y - 2)$
   (h) $(2a + b)(c - 2d)$
   (i) $(x + y)(z - 3)$
   (j) $(2a + b)(3c - d)$
3. (a) $(1 + pq)(1 + p^2)$
   (b) $(p - r)(pq - 2r)$
   (c) $(7x - 1)(7x + a)$
   (d) $(p + 2q)(3p - 4r)$
   (e) $a(2a + 3)(2a^2 + 1)$
   (f) $xy(x + y)(y - 5)$
   (g) $a(5a - 2b + 30)$
   (h) $2x(2x - a - b)$
   (i) $m^2(3m - 1)$
   (j) $p(2m - 5n)$

Exercise 3e

1. (a) $(x + 4)^2$
   (b) $(x + 2)^2$
   (c) $(a + 3)^2$
   (d) $2(x + 1)^2$
   (e) $3(x + 2)^2$
   (f) $4(k + 4)^2$
   (g) $(x - 3)^2$
   (h) $(x - 4)^2$
   (i) $(h - 2)^2$
   (j) $2(x - 1)^2$
2. (a) $(x + y)^2$
   (b) $(x + 3y)^2$
   (c) $(4x + y)^2$
   (d) $(3a + 4b)^2$
   (e) $4(x + y)^2$
   (f) $(5x - y)^2$
   (g) $(7x - 3z)^2$

3. (a) $(x+2)(x-2)$
(b) $(x+4)(x-4)$
(c) $(k+9)(k-9)$
(d) $(5a+8)(5a-8)$
(e) $(6x+7)(6x-7)$
(f) $(9+4x)(9-4x)$
(g) $(8+3a)(8-3a)$
(h) $(9+2h)(9-2h)$
(i) $2(x+3)(x-3)$
(j) $3(x+7)(x-7)$

4. (a) $(h+k)(h-k)$
(b) $(x+4y)(x-4y)$
(c) $(2c+5d)(2c-5d)$
(d) $(6b+a)(6b-a)$
(e) $(7c+3d)(7c-3d)$
(f) $2(x+5y)(x-5y)$
(g) $3(x+3y)(x-3y)$
(h) $4(4a+b)(4a-b)$
(i) $\left(k+\frac{1}{2}h\right)\left(k-\frac{1}{2}h\right)$

5. (a) 1800 (b) 3600
(c) 10 600 (d) 54
(e) 170 (f) 41 200
(g) 795 600 (h) 806 000
(i) 526 000 (j) 318 000

6. (a) 9476 (b) 51.6
(c) 2700 (d) 2300
(e) 470 (f) 840

7. (a) $(x+y+1)(x-y-1)$
(b) $(c+d+2)(c-d-2)$
(c) $a(a+6)$
(d) $-(19+5a)(11+5a)$
(e) $(2x+9)(2x-5)$
(f) $(11+5a)(1-5a)$
(g) $(1+5x)(1-5x)$
(h) $(7a+b+5)(7a-b-5)$
(i) $(2x+2y-1)(2x-2y-1)$
(j) $(5a+b-1)(5a-b+1)$

Exercise 3f

1. (a) $(x+4)(x+2)$
(b) $(y+2)(y+1)$
(c) $(m+8)(m+1)$
(d) $(b+7)(b+4)$
(e) $(x-8)(x-3)$
(f) $(e-2)^2$
(g) $(m-5)(m-4)$
(h) $(x+2)(x-1)$
(i) $(a-2)(a-7)$
(j) $(a+4)(a-2)$

2. (a) $(2x+3)(x+4)$
(b) $(3a+7)(a+1)$
(c) $(4a-3)(a-1)$
(d) $(5p-3)(p-2)$
(e) $(6a-5)(a+4)$
(f) $(5p-3)(p+2)$
(g) $(3p+4)(2p-5)$
(h) $(4a-3)(a-1)$
(i) $(2m+1)(2m+3)$
(j) $(3p-4)(2p+5)$

3. (a) $4(x+1)(x-2)$
(b) $3(x+2)(x+3)$
(c) $2(2x+1)(x+2)$
(d) $3(2x-3)(x+4)$
(e) $4(2x-5)(x+3)$
(f) $2(3x+1)(2x+1)$
(g) $3(3x-2)(2x-3)$
(h) $2(3x-2)(2x+3)$
(i) $2(2x-3)(x-4)$
(j) $5(7x-3)(x+2)$

4. (a) $(6ab+5)(ab-4)$
(b) $(3xy-2)(2xy+3)$
(c) $(5pq+3)(pq-2)$
(d) $(3xy+4)(2xy-5)$
(e) $(4+xy)^2$
(f) $(5-hk)^2$
(g) $(2hk-3)(hk+5)$
(h) $(2mn-1)(2mn-3)$
(i) $2(3pq-4)(2pq+5)$
(j) $(5hk+3)(hk+2)$

Exercise 3g

1. (a) 0 or 9 (b) 0 or –7
(c) 0 or 5 (d) 0 or $-\frac{1}{2}$
(e) 0 or $1\frac{1}{4}$ (f) 0 or $-1\frac{3}{4}$
(g) –2 or –3 (h) –5 or 7
(i) 4 or –11 (j) 4 or 9

2. (a) 0 or –9 (b) 0 or 7
(c) 0 or –4 (d) 0 or –5
(e) 0 or $1\frac{1}{3}$ (f) 0 or $\frac{1}{27}$
(g) 0 or 4 (h) 0 or 3
(i) 0 or –3

3. (a) ±4 (b) $\pm 2\frac{1}{2}$
(c) ±8 (d) ±1
(e) ±5 (f) ±5
(g) $\pm\frac{1}{2}$ (h) $\pm\frac{4}{5}$
(i) $\pm 3\frac{1}{3}$

4. (a) $e = 8$
(b) $d = 3$ or -9
(c) $a = -6$
(d) $q = 5$ or -12
(e) $b = 15$ or -8
(f) $a = \frac{1}{2}$ or $-\frac{1}{5}$
(g) $k = 9$ or -7
(h) $p = 2$ or $1\frac{1}{3}$
(i) $m = \frac{1}{2}$ or -3

5. $k = -8, x = 5$

6. $h = 7, x = 2$

7. $c = -3, x = -\frac{1}{2}$

Exercise 3h

1. 3
2. 2
3. 7, 8
4. 5, 7
5. 8, 10
6. 9, 18 or – 9, –18
7. 4 m
8. 5
9. 6, 9
10. 34 cm
11. 5, 12
12. 4, 9
13. 16 cm, 28 cm
14. 1 or 3

Review Questions 3

1. (a) $6a^2 + 17ab + 12b^2$
(b) $12a^2 - 23ab + 5b^2$
(c) $5x^2 - 13xy - 6y^2$
(d) $7x^2 + 17xy - 12y^2$
(e) $\frac{4}{9}x^2y^2 - 4xy + 9$
(f) $9a^2 + 4\frac{4}{5}ab + \frac{16}{25}b^2$
(g) $\frac{1}{16}a^2 + \frac{1}{12}ab + \frac{1}{36}b^2$
(h) $\frac{1}{16}a^2b^2c^2 - \frac{3}{4}abcx^2yz + 2\frac{1}{4}x^4y^2z^2$
(i) $\frac{9}{16}x^2y^2 - \frac{1}{9}a^2b^2$
(j) $\frac{1}{4}x^2 - \frac{1}{16}y^2$

2. (a) $4(x-2y+4z)$
(b) $5a(a+2b+2c)$
(c) $(a+b)(x-z)$

(d) $(2x + y)(2x + y - 3)$
(e) $(p + q)(3a - 4b)$
(f) $(m - 2n)(5 - m + 2n)$
(g) $(x - y)(a - b)$
(h) $(x - 2y)(x + z)$
(i) $(3a - 2)(a^2 + 1)$
(j) $(x + 1)(x + 2)(x - 2)$

3. (a) $(x + 5)^2$
(b) $(y + 7)^2$
(c) $2(z + 3)^2$
(d) $(x - 4)^2$
(e) $(a - 6)^2$
(f) $3(b - 1)^2$
(g) $6(p^2 + 2q)(p^2 - 2q)$
(h) $2p^2(p - 3q)(p + 3q)$
(i) $x^2y^2(x + 2y)(x - 2y)$
(j) $2x(4y^2 + x^2)(2y + x)(2y - x)$
(k) $4(4a^2 + b^2)(2a + b)(2a - b)$

4. (a) $(x + 9)(x + 4)$
(b) $(a - 1)(a - 19)$
(c) $(a - 1)(a + 16)$
(d) $(2p - 3)^2$
(e) $(5pq + 1)^2$
(f) $(3 + y)^2$
(g) $(1 + 6xy)^2$
(h) $(7 + 2a)(7 - 2a)$
(i) $10(3xy + 1)(3xy - 1)$

5. (a) 2 or 10
(b) 3 or $-\frac{7}{11}$
(c) 3 or $-\frac{3}{5}$
(d) 5 or $\frac{1}{2}$
(e) 0 or 2
(f) 0 or − 3
(g) − 3 or 3
(h) − 2 or 2
(i) − 1 or − 5
(j) 9 or − 1

6. $y = \frac{2}{3x}$

7. (a) $(16x + 2)$ cm
(b) $(15x^2 - x - 6)$ cm²
$x = 4$; 66 cm

8. 5

9. $x = 6$

10. 3 or 12

11. 30

12. $\frac{40}{x}$, $\frac{40}{x-3}$, $\frac{40}{x-3} - \frac{40}{x} = \frac{2}{3}$;
2 h 40 min

13. $\frac{240}{x-4} - \frac{240}{x} = 2$; \$10

14. (a) $\left(\frac{80}{x}\right)$ h
(b) (i) $(x + 5)$ km/h
(ii) $\left(\frac{80}{x + 5}\right)$ h
(c) $\frac{80}{x} - \frac{80}{x + 5} = \frac{1}{4}$
(d) 37.58 or − 42.58
(e) 128 min

15. (a) $9\pi - x^2$
(b) 6.284 cm²

Revision Exercise I No.1

1. (a) 8, \$2 (b) 20

2. (a) 1:800 000
(b) 9 cm
(c) 7.75 cm²

3. (a) 6 cm, 4 cm (b) 20

4. (a) $y - 5x$
(b) $x^2 + 6xy$
(c) $22x^2 - 12xy + 19xy$

5. (a) $\frac{5}{6}$ (b) 189

6. (a) $(x + y)(p - q)$
(b) $(2c - 3)(c - 1)$

7. (a) $\frac{1}{2}$ or -3
(b) $\frac{1}{4}$ or $1\frac{3}{4}$

8. (a) 100 000 (b) 10

9. 189 cm³

10. (a)(i) $10\frac{2}{3}$ (ii) $27\frac{9}{16}$
(b)(i) $3\frac{22}{23}$ (ii) 2.57

Revision Exercise I No. 2

1. (a) 64
(b) \$210, \$60

2. (a) $x^2 - 7x - 8$
(b) $11x - 2$

3. (a) 117°, 141°, 102°
(b) 72

4. (a) $3a^2 + 23ab + 43b^2$
(b) $17a^2 - 2ab$

5. (a) 45 cm (b) 2.88 km²

6. (a) $5xy(x - 3y - 5)$
(b) $(2a - 3b)(x - y)$
(c) $3x(5 + 3y)(5 - 3y)$
(d) $(3x - 2)^2$

7. (a) $2\frac{1}{4}$
(b) (i) 95° (ii) 40°
(c) $\frac{1}{3}$

8. (a) $6^3 - 6 = 210 = 5 \times 6 \times 7$
(b) $18 \times 19 \times 20$
(c) $(x - 1)x(x + 1)$

9. ΔABC and ΔAPQ; $x = 7.8$,
$y = 9\frac{1}{6}$

Revision Exercise I No. 3

1. (a) $x = 78°$, $y = 130°$
(b) 12°

2. (a) 30°, 90° (b) 6.5 cm

3. (a) $2x(3x + 2y)(3x - 2y)$
(b) $2x^2 + 2xy - 2y^2$

4. (a) $3(x + 2)(x - 2)$
(b) 60 km/h

5. (a) $y = 3(2x - 3)$
(i) 33 (ii) 12
(b) $y = \frac{18}{x^2}$; $4\frac{1}{2}$

6. (a) 2480 m (b) 0.75 cm²

7. 6 cm, 9 cm

8. (a) 18 (b) 20
(c) 40

9. (a) 23.79 m³, \$48.53
(b) 8.16%

10. (a) (i) 7 (ii) 1
(b) $5\frac{2}{3}$

Revision Exercise I No. 4

1. (a) \$229.50 (b) 12 km/h
(c) 90 km

2. (a) (i) $2a - b$ (ii) $2x^2 + x - 1$
(b) (i) 40 (ii) 3
(iii) 21

3. (a) $1\frac{1}{7}$ (b) $\frac{2}{5}$
(c) $-\frac{3}{13}$
4. (a) 6, 8 or – 8, – 6
(b) 10 yrs, 40 yrs
5. (a) 6 (b) 165
6. (a) 2 cm (b) 16 m^2
7. (a) 8% increase
(b) 17940 *l*; \$36.60
8. 10.5 cm, 8.3 cm
9. 89.6 km^2
10. (a) $c = 4$; $y = 9t$
(b) $y = \frac{8}{125}t$

Revision Exercise I No. 5

1. (a) 60 cm
(b) 1 h 49 min
2. (a) 6 yrs; 48 yrs
(b) 6 days
3. (a) (i) 4.2 km
(ii) 29.6 cm
(iii) 19.2 cm^2
(b) (i) 11.75 cm
(ii) 51.2 km
(iii) 51.2 km^2
4. (a) $-8x^2 + 10x - 3$
(b) $x^3 + 2x^2 - 5x - 10$
(c) $6 - 5x - 21x^2 - 10x^3$
5. (a) $(x - 5)(x + 2)$
(b) $(x + 2)(x - y)$
(c) $3(3x + 2y^2)(3x - 2y^2)$
6. (a) 9 or – 3
(b) 7 or – 2
7. (a) \$9.20
(b) 126
8. (a) 24.5 cm, 26.25 cm
(b) $x = 16\frac{2}{3}$, $y = 20$
(c) 8.1 cm
9. (a) $-x^2 + 14x - 15$
(b) $x^3 - 23x^2 + 39x - 12$
10. (a) $y = 30$ (b) $x = 64$

Exercise 4a

1. $\frac{3}{4x}$ 2. $\frac{4x^2}{7y^2}$
3. $\frac{b}{3a}$ 4. $\frac{a^2}{3b^2c^2}$
5. $\frac{mp^2}{4n^2}$ 6. $\frac{c^2}{6ab^2}$
7. $\frac{x}{3y}$ 8. $\frac{c^2}{6ab}$
9. $\frac{1}{3(a-b)}$ 10. $\frac{9}{3b(a-3b)}$
11. $\frac{(x+2y)^2}{6x^2}$ 12. $\frac{17}{12a(b+c)^2}$

Exercise 4b

1. $\frac{a+2c}{4}$ 2. $\frac{y}{4}$
3. $\frac{e}{d-e}$ 4. $\frac{1-2d}{3d}$
5. $\frac{3}{2}$ 6. $\frac{1}{4(3c+4d)}$
7. $\frac{c}{d}$ 8. $\frac{3}{4c}$
9. $\frac{a-c}{a}$ 10. $\frac{4}{c}$
11. $\frac{2y}{x}$ 12. $\frac{m}{m-p}$
13. $\frac{1}{a-b}$ 14. $\frac{x+y}{x-y}$
15. $\frac{4}{9}$ 16. $p + 2$
17. $\frac{c+2}{c}$ 18. $\frac{3}{d+3}$
19. $\frac{a-b}{a+b}$ 20. $\frac{a-n}{a+n}$
21. $\frac{q+3}{q+2}$ 22. $\frac{m+3}{m-4}$

Exercise 4c

1. $\frac{3}{2b^4}$ 2. $\frac{3a^3p}{b^3}$
3. $\frac{1}{y^2}$ 4. $\frac{p^3}{8}$
5. $\frac{2p}{7mn}$ 6. $\frac{36ax^2}{7b^2}$
7. $\frac{acd^2}{6b^2}$ 8. $\frac{20a^2b}{9c}$
9. $\frac{27}{64u^3v^3}$ 10. $\frac{8}{5y}$
11. $\frac{xy^3z^2}{2}$ 12. $\frac{6a^2d^3}{b^3c^2}$

Exercise 4d

1. $\frac{3}{4}$ 2. $2x$
3. $\frac{3}{8}$ 4. $\frac{9}{5x}$
5. $\frac{9x}{4y}$ 6. -1
7. $\frac{1}{d(c-d)}$ 8. $b(a - 2b)$
9. $\frac{d+2}{d}$ 10. $\frac{m}{m+3}$
11. $\frac{(b-2c)^2}{c(b+2c)}$ 12. $\frac{-a}{a+2}$
13. $\frac{p(b-2)}{2}$ 14. $\frac{y-2}{3(1-3y)}$

Exercise 4e

1. $\frac{3x-1}{4}$ 2. $\frac{y-x}{3xy}$
3. $\frac{x^2}{4}$ 4. $\frac{3y+2}{5}$
5. $\frac{-7c-18}{15}$ 6. $\frac{e+1}{5}$
7. $\frac{c+3d}{5}$ 8. $\frac{2m-n}{n}$
9. $\frac{a+2x}{3a}$ 10. $\frac{7}{6(e-f)}$
11. $\frac{2(6c+5d)}{15(2c-d)}$ 12. $\frac{5a+7}{6(a-4)}$
13. $\frac{2}{2a+1}$ 14. $\frac{3c}{4(3c+1)}$
15. $\frac{x+y}{2y}$

Exercise 4f

1. (a) $\frac{b-a}{a(a+b)}$ (b) $\frac{2(4a+5)}{a(a+2)}$
(c) $\frac{-x-17}{(x+5)(x-1)}$
(d) $\frac{2(5x-3)}{(x-5)(2x+1)}$
(e) $\frac{15x-41}{(2x-5)(x-3)}$
(f) $\frac{-13x-17}{(3x+2)(2x-7)}$
(g) $\frac{17x}{(3x-5)(4x-1)}$

(h) $\dfrac{5(x-1)}{(5x+1)(5x-1)}$

(i) $\dfrac{15x^2-24x-14}{(6x+7)(5x-4)}$

(j) $\dfrac{52-49x}{(3x-7)(6-5x)}$

2. (a) $\dfrac{2x+5}{x^2-1}$ (c) $\dfrac{26-7x}{x^2-9}$

(c) $\dfrac{2(4-5x)}{4x^2-1}$ (d) $\dfrac{2(9x-5)}{9x^2-4}$

(e) $\dfrac{5(x+3)}{16x^2-25}$ (f) $\dfrac{2x-1}{(x-2)^2}$

(g) $\dfrac{5-2x}{x^2-6x}$

(h) $\dfrac{8x+5}{(2x+1)^2}$ (i) $-\dfrac{1}{x}$

(j) $\dfrac{10(x-2)}{(3x-5)^2}$

3. (a) $\dfrac{a^2+6ab-6b^2}{a^2-b^2}$

(b) $\dfrac{1+37a}{a^2-12a+27}$

(c) $\dfrac{2(2x-3y)}{2x+3y}$

(d) $\dfrac{x^2+3xy-y^2}{y(x+y)(x+y)}$

(e) $\dfrac{6m^2-25m+12}{m(m-3)(m-4)}$

(f) $\dfrac{20m+1}{(2m-1)(m+2)}$

(g) $\dfrac{2a^2-3a+3}{(a-1)^3}$

(h) $\dfrac{13-5a}{(a-1)(a-2)(a-3)}$

(i) $\dfrac{-2}{a+1}$

(j) $\dfrac{3}{3+2a}$

Exercise 4g

1. (a) $x=4$ (b) $m=3$
   (c) $x=3\frac{1}{2}$ (d) $x=5\frac{3}{5}$
   (e) $p=120$ (f) $e=3\frac{3}{4}$
   (g) $e=6$ (h) $a=6$
   (i) $x=9$ (j) $x=11\frac{1}{2}$
2. (a) $m=-3$ (b) $x=1\frac{1}{3}$
   (c) $m=6$ (d) $a=5\frac{4}{5}$
   (e) $x=\frac{2}{3}$ (f) $x=-\frac{12}{13}$
   (g) $a=-\frac{6}{7}$ (h) $d=2$
   (i) $x=-\frac{1}{2}$ (j) $x=\frac{11}{42}$

Exercise 4h

1. 5
2. 8
3. $\frac{3}{5}$
4. 2
5. 7
6. 15, 24
7. 24, 45
8. 7 yr
9. 27 yr
10. 22 yr
11. 20 cm
12. 20 km
13. 120 km
14. 336 km
15. 288 sweets
16. 672 matches
17. (a) $\dfrac{420}{x}$ h
    (b) $\dfrac{420}{x+15}$ h
    (c) $\dfrac{420}{x}-\dfrac{420}{x+15}=\dfrac{40}{60}$
    (d) $x=90$; 4 h 40 min
18. 48 km/h
19. $x=12$
20. $x=6$, 32 h

Exercise 4i

1. (a) $a=\dfrac{y}{x}$ (b) $a=\dfrac{q}{p-4}$
   (c) $a=\dfrac{c-by}{x}$ (d) $a=\dfrac{c}{p}-b$
   (e) $a=\dfrac{7-3m}{2}$
   (f) $a=\dfrac{5b-3c}{2}$
   (g) $a=mc-mb=m(c-b)$
   (h) $a=\dfrac{3x-15z}{2}=\dfrac{3(x-5z)}{2}$
   (i) $a=14p$ (j) $a=\dfrac{R}{m}-g$
2. (a) $b=\dfrac{2A}{h}$ (b) $k=\dfrac{2Tl}{x^2}$
   (c) $t=\dfrac{m(v-u)}{F}$
   (d) $l=\dfrac{15k-12}{2}$
   (e) $a=\dfrac{bc}{c-b}$
   (f) $a=\dfrac{3}{5}c-\dfrac{3}{4}b$
   (g) $a=\dfrac{2A}{h}-b$
   (h) $l=\dfrac{2s}{n}-a$
   (i) $q=\dfrac{fp}{p-f}$
   (j) $u=\dfrac{s}{t}-\dfrac{1}{2}gt$
3. (a) $l=\dfrac{1}{2}p-b$
   (b) $h=\dfrac{A}{2\pi r}-r$
   (c) $c=\dfrac{3a}{a-b}$
   (d) $a=\dfrac{bc}{x-c}$
   (e) $b=\dfrac{a}{a-1}$
   (f) $b=\dfrac{4}{5}a+18$
   (g) $b=\dfrac{ac}{a-c}$
   (h) $a=\dfrac{c}{2}(4-b)$
   (i) $a=b^2-b$
   (j) $z=\dfrac{y^2}{y-x}$
   (k) $x=\dfrac{bc}{a+c}$
   (l) $y=\dfrac{3-x}{x-2}$

Exercise 4j

1. (a) $a = b^2$ (b) $a = \frac{1}{2}b^2$
   (c) $a = \frac{1}{5}(e^2 + 8)$
   (d) $a = 2b^2$
   (e) $a = \frac{5cx^2}{2}$ (f) $a = \frac{2b}{3b - 1}$
   (g) $a = \pm\sqrt{\frac{b-3}{2}}$
   (h) $a = \pm\sqrt{\frac{A}{4\pi}}$
   (i) $a = b + c^3$
   (j) $a = \pm\sqrt{\frac{2Knb}{m}}$
2. (a) $b = \frac{1}{2}(a^2 - a)$
   (b) $y = \pm\sqrt{x} - x$
   (c) $w = \pm\sqrt{\frac{x-b}{2}}$
   (d) $b = \pm y\sqrt{1 - \left(\frac{a}{x}\right)^2}$
   (e) $x = \frac{t^2(m-3)}{4}$
   (f) $c = \frac{ab + 2b - 2a}{b - a}$
   (g) $g = \frac{3x}{12kt + 8m}$
   (h) $b = \pm\sqrt{D^2 + 4ac}$
   (i) $s = \pm\sqrt{\frac{5p}{2p - 3q}}$
   (j) $h = \frac{2kq^2 - 9p^2n^2k}{27p^2n^2 - q^2}$
3. (a) $\frac{q^2}{q+1}$
   (b) $a = 2mk - 2m^2l^2$
   (c) $r = \frac{wr}{a + w}$
   (d) $x = \frac{3 - y}{2y + 1}$
   (e) $c = \frac{4}{1 + a}$
   (f) $q = \frac{pr - m}{r}$
   (g) $p = 3q$
   (h) $x = \frac{10 - 2b}{2 + 3b}$

Exercise 4k

1. 45
2. ±4
3. (a) $\pm 2\frac{2}{5}$ (b) $-140\frac{1}{6}$
4. $\pm\sqrt{\frac{23}{5}}$ (b) $c = 6\frac{15}{17}$
5. (a) 4 (b) ±10
6. $5\frac{5}{7}$ (b) 126
   (c) $\pm\sqrt{132}$
7. (a) 2206.7 (b) 19.07
8. (a) 7861.3 (b) 16.66
   (c) 6.441 (d) 9.81
9. (a) 17.60 (b) 688.4
   (c) ±27.49

Review Questions 4

1. (a) $\frac{4}{3x}$ (b) $\frac{x-3}{9}$
   (c) $\frac{11x + 13}{30}$ (d) $\frac{11}{4x}$
   (e) $-\frac{2}{3x}$ (f) $\frac{4 - 3x}{5(2x - 1)}$
   (g) $\frac{3x^2 + 2}{x}$ (h) $\frac{9x + 18}{5}$
   (i) $\frac{4x + 5}{(2x - 1)(5x + 1)}$
   (j) $\frac{7x + 10}{(x + 1)(3x + 4)}$
   (k) $\frac{8x - 17}{(x - 2)(3x - 7)}$
   (l) $\frac{x + 5y}{(y + x)(y - x)}$
   (m) $\frac{2(8x + 1)}{(2x + 1)(2x - 1)}$
   (n) $\frac{3y - x}{x(x - y)}$ (o) $\frac{17x + 6y}{4x(x - 2y)}$
2. (a) $2\frac{4}{5}$ (b) $14\frac{2}{3}$ (c) 7
   (d) $2\frac{1}{7}$ (e) $\frac{1}{10}$ (f) −37
   (g) 10 (h) 35 (i) $-2\frac{3}{4}$
   (j) $2\frac{1}{7}$ (k) −1 (l) $\frac{7}{20}$
3. (a) $y = \frac{x}{1 - x}$ (b) $t = \frac{1 - a}{1 + a}$
   (c) $a = \frac{b}{1 - c}$ (d) $a = \frac{b}{x - 3}$
   (e) $p = \frac{100a}{100 + rt}$
   (f) $h = \frac{5k}{5 + 2k}$
4. 16
5. 16
6. 4, 6
7. 4, 12
8. $a = \pm\sqrt{\frac{5t^2}{2\pi^2 r^2} - b^2}$
9. $\frac{acd}{ac - cd - ad}$
10. $y = x^4$
11. 252 km
12. 135 km
13. $12.35
14. $\frac{xy}{x + y}$
15. 36 sweets
16. 48
17. $\frac{67200}{x-5} - \frac{67200}{x} = 32$
18. (a) $\frac{5800}{x}$ (b) $\frac{5800}{x + 12}$
    (c) $\frac{5800}{x} - \frac{5800}{x + 12} = 4\frac{11}{16}$
    (d) $x = 116$; 293 l

Exercise 5a

1. (a) (1, 2) (b) (2, 1)
   (c) (1, 1) (d) (3, 1)
   (e) (1, 3) (f) (−1, 2)
   (g) (1, 2) (h) (3, −1)
   (i) (−2, 4) (j) (−3, 2)
2. (a) (4, 1) (b) (2, 3)
   (c) (5, −1) (d) $\left(2, \frac{1}{2}\right)$
   (e) $\left(1, -\frac{1}{2}\right)$ (f) $\left(2, 1\frac{1}{2}\right)$
   (g) $\left(1\frac{1}{2}, \frac{1}{2}\right)$ (h) $\left(1\frac{1}{2}, -1\right)$
   (i) (1, −1) (j) $\left(1, \frac{1}{4}\right)$
3. (a) (8, 8) (b) (12, 7)
   (c) (9, 3) (d) (7, 9)
   (e) (3, 2) (f) $\left(3, 1\frac{1}{2}\right)$

(g) (2, 1) (h) $\left(2, -1\frac{1}{2}\right)$

(i) (4, 1) (j) $\left(1\frac{1}{2}, -1\frac{1}{2}\right)$

4. (a) (–7, –13) (b) $\left(1, 2\frac{2}{3}\right)$

(c) $\left(5, \frac{2}{5}\right)$ (d) $\left(3\frac{1}{3}, -3\right)$

(e) $\left(1\frac{6}{7}, 10\right)$ (f) (–3, 5)

(g) $\left(-43\frac{1}{2}, -34\right)$

(h) (–5, 6)

Exercise 5b

1. (a) (2, 1) (b) (3, –1)
(c) (–1, 2) (d) (5, –2)
(e) (4, 5) (f) (–1, –2)
(g) (–2, –4) (h) (2, –1)
(i) (3, 2) (j) (4, –5)

2. (a) (1, 2) (b) (2, 3)
(c) (3, 1) (d) (3, –1)
(e) (5, –6) (f) (1, –2)
(g) (1, –3) (h) (2, –2)
(i) $\left(2, -1\frac{1}{2}\right)$
(j) $\left(\frac{1}{2}, 1\frac{1}{2}\right)$

3. (a) $x = 3, y = 7$
(b) $x = 1, y = -1$
(c) $x = 9, y = 4$
(d) $x = 6, y = -4$
(e) $x = 4, y = 9$
(f) $x = 13, y = 11$
(g) $x = -\frac{1}{2}, y = 3$
(h) $x = 16, y = -4$
(i) $x = 12, y = -2$
(j) $x = -1, y = 1$

Exercise 5c

1. (a) (6, 1) (b) (1, 3)
(c) (–1, –1) (d) (3, 10)
(e) (1, 2) (f) $\left(3\frac{1}{7}, 5\frac{6}{7}\right)$
(g) (9, 4) (h) $\left(0, 1\frac{1}{2}\right)$
(i) (2, –1) (j) (–4, 4)

2. (a) (3, 5) (b) $\left(2\frac{1}{2}, 4\right)$
(c) (4, –3) (d) (–3, 5)
(e) $\left(\frac{2}{3}, 0\right)$ (f) (15, –5)
(g) (–2, 11) (h) (3, 6)

Exercise 5d

1. 113, 25
2. 146°, 34°
3. $22\frac{1}{2}, 13\frac{1}{2}$
4. 74°, 46°
5. 1 kg of potatoes costs \$2, 1 kg of carrots costs \$2.40
6. \$3 per stool, \$10 per chair
7. $\frac{7}{9}$
8. \$15 per belt, \$27 per wallet
9. 21 cm
10. $\frac{3}{5}$
11. 40, 8
12. \$32, \$48
13. 30 cm
14. $\frac{47}{7}, \frac{48}{7}$
15. (a) 8 cm (b) 32 cm
16. 9 five-cent coins, 6 twenty-cent coins
17. Father: 40 yr
Son: 10 yr
18. 65
19. 15, 5
20. $x = 6, y = 7$, 44 cm
21. 26, 14
22. 20 fifty-cent stamps, 16 ten-cent stamps
23. $L = 20$ cm, $B = 18$ cm
24. Bank A = \$1 300,
Bank B = \$1 200
25. 420 km
26. $x = 14, y = 7$
Area = 875 cm^2

Review Questions 5

1. (a) $x = 2, y = -2$
(b) $x = 2, y = 2.5$
(c) $x = 1, y = -4$
(d) $x = 1\frac{1}{3}, y = -8$
(e) $x = \frac{3}{4}, y = -\frac{1}{4}$
(f) $x = 3, y = -4$
(g) $x = 1, y = 2$
(h) $x = 3, y = 5$
(i) $x = 2, y = 1.5$

2. apple – 40¢.
orange – 35¢
3. 17, 14
4. A = \$11, B = \$13
5. 6, 14
6. pear – 15¢,
mango – 25¢
7. 40, 110
8. $\frac{7}{10}$
9. $\frac{3}{7}$
10. 11 yr; 41 yr
11. 14 kg, 6 kg
12. 68 km/h, 80 km/h
13. 48 km/h, 52 km/h
14. 48
15. (a) (2, 4)
(b) (–1, 3)
16. there are 14 more hens

Exercise 6a

1. (a) 9.43 (b) 13.1
(c) 13.3 (d) 4.82

2. (a) 15 cm (b) 17 m
(c) 5 m (d) 24 m
(e) 9 cm (f) 29 cm
(g) 12 cm (h) 60 cm
(i) 45 cm (j) 65 cm
(k) 8.06 m (l) 9.22 m
(m) 6.71 cm (n) 8.67 cm
(o) 8.20 cm (p) 12.8 cm

3. (a), (c), (d), (e), (g), (h)

4. (a) 7.07 (b) 7.21
(c) 14.7 (d) 12.3
(e) 20.8

5. (a) $x = 16, y = 9$
(b) $x = 25, y = 144$

6. (a) 22.45
(b) $y = 10, x = 18.03$
$x = 14.73, y = 12$
(d) $x = 17, y = 25.92$
(e) $x = 15.81, y = 12.25$
(f) $x = 8.06, y = 9, z = 9.85, k = 10.63$
(g) $x = 6.02$

Exercise 6b

1. 4.66 m
2. 50.3 m
3. 3.80 m
4. 20.3 cm
5. 10 cm
6. 3
7. 28.3 cm
8. 58.3 m
9. 6.63 m
10. 11.1 sec
11. 1.73 cm
12. 10 cm
13. 6.32 cm
14. 9.22 cm, 5.86 cm
15. 60 cm^2
16. 24 cm, 168 cm^2
17. (a) $6^2 + 8^2$
$= 36 + 64$
$= 100$
$= 10^2$
(b) (i) 12, 16, 20
(ii) 15, 20, 25
(c) (i) $25n^2$
(ii) 21, 28, 35
(d) (i) 7, 24, 25
(ii) No
(iii) 9, 40, 41

Review Questions 6

1. (a) 11.4 (b) 10.3 (c) 9.62
(d) 10 (e) 12 (f) 25
2. 28.7 cm
3. 120 cm; 903 cm^2
4. 92.3 cm, 2210 cm^2
5. 11.2 cm
6. 14.4 cm
7. 11 $units^2$
8. Yes

Revision Exercise II No. 1

1. 308 cm^2
2. (a) 3 or –12 (b) –3
3. $\frac{1}{250\,000}$
4. (a) $2pq - 4p^2 - 7q^2$
(b) $-2ab^2 - 4a^2 + 3b$
5. (a) $5(x - 1)(x - 4)$
(b) $(5a - 3b)(2a - 3b)$
(c) $(7a + 8b)(4a - 3b)$
(d) $(ab + 5)(ab - 5)$
6. (a) $x = \frac{10ay + 4}{15a + 3}$; 3
(b) $a = \frac{3xb + b}{6x - 1}$
(c) $x = \frac{yk}{n + yh}$
7. $x = 2, y = -3$
8. $x = 5\frac{5}{7}$; 8.06, 3.46
9. (a) $\frac{28a + 22b}{15}$
(b) $\frac{9 + 14a}{42a}$
10. $\frac{6}{7}$

Revision Exercise II No. 2

1. 1 min
2. (a) $a^3 + b^3$
(b) $x^3 - y^3$
3. (a) 29, 40
(b) 58, 94
(c) $\frac{2}{13}, \frac{1}{14}$
4. (a) $3\frac{3}{5}$
(b) $8\frac{1}{2}$
(c) 1 or $-\frac{1}{3}$
(d) 2 or –9
5. (a) – 2 (b) $2\sqrt{3}$
6. (a) $x = \frac{3ab}{2a - b}$
(b) $a = \frac{4c - 5F}{2b - 3F}$
7. $y = 24, x = 4, y = 3x^3$
8. (a) $x = 3, y = -1$
(b) (i) \$28
(ii) \$15
9. (a) $(2x + 5)(2x + 7)$
(b) $(5x + 3)(x - 1)$
(c) $(2x - 3)(3x - 7)$
10. $x = 10.8, y = 19.2.\ z = 14.4$

Revision Exercise II No. 3

1. (a) $r = \frac{S - a}{S}$; $\frac{1}{10}$
(b) $c = \frac{2ab}{3(2b - a)}$; $\frac{5}{6}$
2. (a) 65.71 km/h
(b) 56 cm
3. \$4.80
4. (a) $4\frac{1}{2}$ (b) $1\frac{1}{10}$
5. 30¢, 25¢
6. $x = 11$; 7 cm, 24 cm, 25 cm
7. (a) $r = 15$ (b) $F = 26\frac{2}{3}$
8. (a) $(x - 15)(x - 23)$
(b) $(2x - 5)(x + 1)$
(c) $(3x + 4y)(3x - 4y)$
9. (a) 0 or –3
(b) $-\frac{2}{3}$ or –3
10. (a) $5x^2 + 30x - 10$
(b) $\frac{2x}{(x - 1)(x + 1)}$
(c) $\frac{x^2 + 2x + 2}{1 - x^2}$

Revision Exercise II No. 4

1. (a) $\frac{18}{25}$ (b) $\frac{1}{4}$
2. (a) 549
(b) $m = \frac{2E}{V^2 + 2gh}$
3. 668 cm^2, 103.7 cm
4. (a) 5.97 s
(b) 30.5 cm
5. (a) $2(x + 2)(x + 3)$
(b) $(x + y + 2)(x + y + 3)$
(c) $(a^2 + b^2)(a + b)(a - b)$
6. $x = 5, y = 3$
7. (a) $x = \frac{abc}{c - a}$
(b) $x = \frac{b + d}{c - a}$
8. $\frac{5}{7}$
9. (b) 18.8%
(c) 0.123%, 6.99%
(d) Asian economic crises

Revision Exercise II No. 5

1. (a) 2.29%
(b) 71.10 hours
2. (a) 42 kg
(b) 67, 9

3. $k = \frac{A}{R^2 - r^2}$; 3.125

4. 32.3 cm, 6.32 cm

5. $x = 0, y = -1$

6. 875 cm^2, 43.01 cm

7. (a) $t = \frac{v - u}{a}$, 2.5

   (b) 15, 16

8. (a) $\frac{-x - 12}{(x + 2)(x - 3)}$

   (b) $\frac{-22x}{x^2 - 36}$

   (c) $\frac{6 - 5x}{2x(x + 6)}$

9. (a) $x = \frac{a}{b + 1}$

   (b) $x = \frac{b + ab}{ac - a}$

10. (a) (i) $(x^2 + 2)(x + 1)(x - 1)$

    (ii) $(x+2)(x-2)(x+3)(x-3)$

    (b) (i) 243 (ii) 391

Exercise 7a

1. 20 cm^3
2. 46 cm^3
3. 2 550 000m^3
4. 23.3 cm^3
5. 5 cm
6. 9 m
7. 7.5 cm
8. 5 cm
9. 960 cm^3
10. 32.5 cm
11. 0.75 cm
12. 2670 g
14. (a) 13.7 cm
    (b) 660 cm^3
    (c) 504 cm^2
15. 7824 cm^2, 53.7 cm^3, 37780.6 cm^3
16. 173 cm^2
17. 6.93 cm
18. 8 cm
19. 12 m

Exercise 7b

1. (a) 528 cm^3
   (b) 257 cm^3
   (c) 180 cm^3
   (d) 12 941 mm^3
2. 15 cm
3. 24 m
4. 3.00 cm
5. 28.7 cm
6. 8192
7. (a) 88.0 cm
   (b) 1320 mm^2
   (c) 1650 cm^2
8. 9.01 cm
9. 14 mm
10. 16.0 cm
12. (a) 2.5 cm
    (b) 39.3 cm^2
13. 204 cm^2
14. 1230 cm^3
15. 2706 mm^3, 1291 mm^2
16. (a) 352 m^2
    (b) 553 m^3
17. (a) 603 cm^3
    (b) 377 cm^2
18. 1560 cm^3

Exercise 7c

1. (a) 2140 cm^3
   (b) 11 500 mm^3
   (c) 268 m^3
2. 1890 g
3. 709
4. 621 kg
5. (a) 6.97 cm
   (b) 14.3 m
   (c) 5.71 m
6. (a) 9 cm
   (b) 7.20 mm
   (c) 2.25 m
7. 3 cm
8. (a) 1810 cm^2
   (b) 1020 mm^2
   (c) 113 m^2
9. 462 cm^2
10. 5.88m, 434 m^2
11. (a) 4.09 cm
    (b) 24.0 mm
    (c) 15.9 m
12. (a) 4 cm
    (b) 15.1 mm
    (c) 3.5 m
13. 12.0 cm, 7240 cm^3
14. (a) 2258 cm^3, 902 cm^2
    (b) 32 340 cm^3, 5082 cm^2
15. 313.45 m^3
16. (a) 5088π cm^3
    (b) 1104 cm^2
17. (a) 15 300 cm^3
    (b) 3210 cm^2
18. (a) 44 400 cm^3
    (b) 7610 cm^2
19. 1.28 cm
20. 20.3 cm
21. 14.9 cm
22. (a) 182 cm^2
    (b) 2.27 cm

Review Questions 7

1. (a) 134 000 cm^3, 15 100 cm^2
   (b) 334 m^3, 242 m^2
   (c) 2.59 m^3, 11.9 m^2
   (d) 23 200 cm^3, 5440 cm^2
   (e) 13 100 cm^3, 3600 cm^2
2. (a) 323.87 cm
   (b) 523.67 cm^2
   (c) 100.54 cm^2
3. (a) 4 m
   (b) (i) 0.483 m^2
   (ii) 5.79 l
   (c) 115 h
4. $235.62
5. $141.81
6. $8\frac{8}{9}$ cm
7. 1020 cm^2
8. 5.34 cm, 3.44 kg
9. (a) 400
   (b) $\sqrt[3]{\frac{3V}{4\pi}}$, 1.44 mm
10. 3 : 1 : 2
11. 113 150 litres, 43 986
12. 8062.65 m^3, 117 253
14. 10 115 kg/m^3
15. (a) 794 g
    (b) 10 525.76 kg/m^3
16. 220 cm
17. 7.93 cm
18. (a) 0.6 cm
    (b) 0.622 cm^3, 0.679 cm^3

Exercise 8a

1. (a) 0, 1, 2, 3, 4
   (b) 1, 2, 3, 4, 5
   (c) 2, 0, – 2, – 4, – 6
   (d) –5, – 4, –3, –2, –1
3. (a) 2, 5, 8
   (b) (i) –1, 6.8, 9.5
   (ii) –2, –1.4, – 0.7
4. (a) – 4, 0,4
   (c) (i) –2, 6, 10
   (ii) – 0.5, 0.4, 0.9

Exercise 8b

1. (a) $y = 2$ (b) $y = 9$
   (c) $y = -3$ (d) $y = -\frac{1}{2}$
   (e) $y = 0$
2. (a) $x = 12$ (b) $x = 5$
   (c) $x = -4$ (d) $x = 0$
   (e) $x = -\frac{1}{4}$
3. (a) $y = -6$ (b) $x = 3$
   (c) $x = -10$ (d) $y = 8$
   (e) $y = -\frac{1}{3}$
4. $y = 15, y = 11, y = 3, y = 0, y = -3, y = -10, y = -16$
5. $x = -18, x = -13, x = -7, x = -3, x = 0, x = 4, x = 11, x = 14, x = 18$
6. (a) $\left(-1, 3\frac{1}{2}\right), \left(0, 3\frac{1}{2}\right), \left(1, 3\frac{1}{2}\right), \left(2, 3\frac{1}{2}\right)$
   (b) (–6, –1), (–6, 0), (–6, 1), (–6, 2)
   (c) $\left(\frac{2}{5}, -1\right), \left(\frac{2}{5}, 0\right), \left(\frac{2}{5}, 1\right), \left(\frac{2}{5}, 2\right)$
   (d) (–1, –7), (0, –7), (1, –7), (2, –7)
   (e) (–1, 4.2), (0, 4.2), (1, 4.2), (2, 4.2)
   (f) (–3.3, –1), (–3.3, 0), (–3.3,1), (–3.3, 2)
   (g) (20, –1), (20, 0), (20, 1), (20, 2)
   (h) (–1, –16), (0, –16), (1, –16), (2, –16)

Exercise 8c

1. (a) $\frac{3}{2}$ (b) $-\frac{5}{3}$ (c) $\frac{5}{3}$
   (d) $-\frac{3}{2}$ (e) $-\frac{1}{4}$ (f) $-\frac{1}{2}$
   (g) 4 (h) $-\frac{1}{3}$
2. Yes, (2, 2), (4, 2), (4, 6), (6, 6)
3. Square; (4, 2), (2, 4), (6, 4), (4, 6)
4. Rhombus; (6, 2), (12, 4), (0, 4), (6, 6)
5. Kite; (2, 6), (6, 0), (6, 8), (10, 6)

Exercise 8d

1. $x = -1, y = -3$
2. $x = 1, y = -1$
3. $x = -5, y = -2$
4. no solutions
5. infinite number of solutions
6. $x = 3, y = 1$
7. $x = 4, y = 2$
8. $x = 0, y = 2$
9. $x = 2, y = -1$
10. $x = 5, y = 3$
11. $x = 2.6, y = -2.6$
12. $x = 3, y = -4$

Review Questions 8

2. (a) 5 (b) 2
   (c) –2 (d) 4
3. (a) 2, 3, –1 (b) 7, 10, 13
   (c) $x = -4, y = 1$
4. $x = 2.5, y = 2.5x, y = -x, x = -3, y = 2.5x$
5. Trapezium; (2, 6) , (4, 8) , (4, 2) , (12, 10)
   (a) –2 (b) 1
   (c) $\frac{1}{4}$ (d) 1
6. (i) (3, 2), $\left(4\frac{4}{5}, 3\frac{1}{5}\right)$, (6, 2); $1\frac{4}{5}$ unit^2
7. (i) (4.4), (0.0), (– 2.4)
   (ii) 12
8. (a) They have equal gradients but different y-intercepts. They are parallel.
   (b) They have equal gradients and equal y-intercepts. They are identical.

Exercise 9a

1. 13, 5, 4, 5, 8, 20
   (a) (0, 4)
   (b) $x = 0$
2. – 14, –5, 1, – 5, – 9
   Yes, $x = \frac{3}{4}$, $\left(\frac{3}{4}, 1\frac{1}{8}\right)$
3. $1\frac{3}{4}, 0, -2\frac{1}{4}, -2, 0$
   (a) $y = -2\frac{1}{4}, x = 2\frac{1}{2}$
   (b) $x = 2\frac{1}{2}$
4. (a) –2, 2, 4, – 8, –16
   (c) (i) – 4.2, 1.2; – 5, 2; 2.7
   (ii) – 4.8, 1.4, – 5.4
5. (a) 3, 0, 3, 8
   (c) (i) 0.3, 1.7; – 0.7, 2.7; –1.4, 3.4
   (ii) 6.8; 0.4; 5.3
   (d) $x = 1$
6. (a) 6, 0, – 4, – 6, – 6, – 4, 6
   (c) (i) – 3.6, 0.61; – 4.9, 1.9; –5.3
   (ii) 9.8, – 4.8, 3.4
   (iii) – 6.25
7. (a) min, (2, – 1)
   (b) min, $\left(-\frac{1}{2}, -5\frac{3}{4}\right)$
   (c) max, (1, – 2)
   (d) max, $\left(\frac{2}{3}, 26\frac{1}{3}\right)$

Exercise. 9b

1. (a) $A = 4x - 4x^2$
   (c) $x = 0.5$
   (d) 0.765 or 0.235
2. (a) $A = 3x^2 - 3x + 2$
   (c) 1.25 cm^2
   (d) 1.62
3. (a) $y = x(64 - 8x)$
   (c) $4
4. (b) (i) 17.2m (ii) 3.8s
5. (b) (i) 8.5 m/s (ii) 2 m/s
   (iii) 0.6s or 3.4s

6. (b) (i) 8 handbags
(ii) 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 handbags.

Review Question 9

1. 4, 6, 4, 0, (a) $(2\frac{1}{2}, 6\frac{1}{4})$
(b) $x = 2\frac{1}{2}$
2. (a) 5, – 4, 0, 12, 21
(c) (i) –3.8, –1.8, 8.3
(ii) ±1.4, 3.2, 3.5
(iii) (0, -4)
3. (a) 16, 9, 1, 0, 9
(c) (i) – 0.7, 2.7; – 1.8, 3.8; – 2.7, 4.7
(ii) 11.6, 0.6, 7.3
(iii) (1, 0)
(iv) $x = 1$
4. (b) (i) – 3.9, 2.9; –2.8, 1.8; – 2, 1
(ii) 7.4, 5.8, 1.3
5. (b) (i) – 0.7, 2.7; – 1.4, 3.4; – 1.8, 3.8
(ii) 0.3, – 5.4, – 3.4
(iii) $y = -6, x = 1$
6. (a) 2.4, 10.7
(b) 8.5
(c) $y = 8.5, x = 6.5$
7. (a) $A = 4 + 3x - x^2$
(c) 1.5
8. (a) $y = x(85 - 6x)$
(c) $7.08 (d) $261
9. (b) (i) 45 items
(ii) $2025
10. (a) 2.2s
(b) 4.4s 30.25 m
11. (–2, 0), (4, 0), (0, –8), (1, –9)
12. (1, 0), (–5, 0), (0, 5), (–2, 9)

Revision Exercise III No. 1

1. (a) 4 (b) 27
(c) $6\frac{1}{2}$ (d) $-\frac{1}{5}$
2. (a) 15yr, 45 yr
(b) 27, 63
3. (a) 226.8 cm^3
(b) 11.03 cm
4. 42.6 cm, 2300 kg
5. (a) $d = \frac{L-N}{L}$; $\frac{1}{5}$
(b) 56
6. (a) 5 cm (b) 30 cm^2
7. 33.6 m
8. – 4, 6, 0
(a) 6.1 (b) 2.7 or – 2.2
9. (3, 2)
10. 5, 0, – 3, 12
$x = 1$; 3 or – 1

Revision Exercise III No. 2

1. (a) – 6 (b) $-2\frac{1}{2}$
(c) 1 (d) 3
2. (a) $1\frac{2}{5}$ (b) $15
3. (a) (6, 0), $B(0, 4)$; 18 $units^2$
(b) $a = 3, b = -4$
4. (a) $S = \frac{v^2 - u^2}{2a}$, 5
(b) 40, 54
5. $5a(2a + 3)(2a - 3)$
6. (a) 405 cm^3 (b) 4.33 cm
7. (a) 6283 cm^3, 1885 cm^2
(b) 1150 cm^3, 531 cm^2
8. (a) 4.9 (b) 2.3 or – 2.3
9. (a) 1 : 75 000
(b) 17.25 km (c) 13.5 km^2
10. – 5, 0, 3, 0
(a) 4 (b) $x = 1$
(c) 3 or –1 (d) 3.45 or –1.45

Revision Exercise III No. 3

1. $C = \frac{3m-b}{2}$
2. (a) ΔABD and ΔCBD, rotate 180° about 0
(b) ΔCPQ and ΔCDB, an enlargement, centre C, scale factor 2.
3. $x = -\frac{1}{2}$, $y = 2$
4. (a) $\frac{13x-5}{15}$ (b) $\frac{2-x}{x(x+2)}$
5. (a) 9
(b) $289 .00
6. (a) 160.8 cm^3, 217.0 cm^2
(b) 50 cm
7. (a) $(6x + 7y)(6x - 7y)$
(b) $(12a + 5)(a - 3)$
(c) $(h + 6k)(h - 9k)$
8. 12 cm, 9.6 cm
9. – 5, – 2, 37
(a) 2.9 (b) 1 or 3.5
10. (a) 13.5 cm (b) 2464 cm^2
(c) 9 cm

Revision Exercise III No. 4

1. (a) $6\frac{2}{9}$ (b) 33
2. $(a + b + c)(a + b - c)$
3. (a) $\frac{x}{21}$ (b) $\frac{x^2+2}{x(x-2)}$
4. $x = 5, y = 1$
5. (b) 9.33 cm
6. (a) 125 cm^3 (b) 60 cm
7. (a) $k = 8\frac{1}{3}$; (0, -5)
(b) (3, -2)
8. (a) 51.3 cm^3 (b) 5.32 cm
9. (a) 1 m (b) 5 cm
10. –4, 2, 0
(a) 2.3 (b) 1.3
(c) – 0.6 or 3.6

Revision Exercise III No. 5

1. (a) 4.27 cm (b) 28.3 cm
2. (a) $x = 32°, y = 48°$
(b) 154 cm^2, 88 cm
3. $x = 3, y = -7$
4. 11
5. (a) 88 cm (b) 484 cm^2
6. (a) 0, 4 or $-\frac{2}{3}$
(b) –1 or $2\frac{2}{3}$
7. (a) 4.9 m (b) 11.2 cm
8. 11, –5, –4, 4
(a) –4.8 (b) 4.3 or –2.3
9. 13 mm
10. $x = 1.6, y = 5.2$

Exercise 10a

1. (a) {January, June, July}
(b) {11, 13, 15, 17}
(c) {b, c, d, f, g}
(d) {Tuesday, Thursday}
(e) {2, 4, 6, 8, 10, 12}
(f) {February}
2. (a) The set of even numbers ≤ 10.
(b) The set of even numbers.
(c) The set of first 5 English alphabet.
(d) The set of ball games.
(e) The set of fruits.

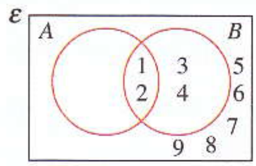
3. (a) T (b) F
(c) F (d) T
4. (a) T (b) T
(c) F (d) F
(e) F (f) F
5. (a) Yes (b) No
(c) No (d) No
(e) Yes (f) No
6. (a) China; the set of ASEAN countries
(b) Rubber; the set of edible fruits
(c) 20; the set of perfect squares
(d) 75; the set of perfect cubes
(e) pie chart; the set of statistical averages
7. (a) {A, E, I}
(b) {red, orange, yellow, green, blue, indigo, violet}
(c) {9, 18, 27, 36, 45}
(e) {New Year's Day, Hari Raya Haji, Chinese New Year, Good Friday, Labour Day, Vesak Day, National Day, Deepavali, Hari Raya Puasa, Christmas Day}
(f) {12, 14, 16, 18, 20, 22}

Exercise 10b

1. (a) 12 (b) 8
(d) 0 (f) 7
(g) 12
2. (a), (d), (f), (h), (i), (l), (m), (n), (p), (q)

Exercise 10c

1. (a) {2, 4, 6, 8, 10, ..., 20}
(b) {4, 8, 12, 16, 20}
(c) {3, 6, 9, 12, 15, 18}
(d) {1, 3, 5, 7, 9, 11, 13, 15, 17, 19}
(e) {1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 18, 19}
(f) {1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20}
2. $A' = \{30, 31, 32, 34, 35, 36, 38, 40, 41, 43, 45\}$
$B' = \{35, 43, 44\}$
$C' = \{31, 37, 41, 43\}$
$D' = \{30, 31, 33, 34, 35, 37, 38, 39, 41, 42, 43, 45\}$
$E' = \{31, 32, 34, 35, 37, 38, 40, 41, 43, 44\}$
3. (a) T (b) T
(c) T (d) F
(e) T
4. $A = \{2, 3, 5, 7\}$,
$B = \{2, 4, 6, 8, 10\}$,
$C = \{5, 10\}$,
$A' = \{1, 4, 6, 8, 9, 10\}$,
$B' = \{1, 3, 5, 7, 9\}$,
$C' = \{1, 2, 3, 4, 6, 7, 8, 9\}$
5. (a) $\neq$ (b) $\supset$
(c) $=$ (d) $\neq$
(e) $\neq$ (f) $=$
(g) $\subset$ (h) $=$
(i) $\neq$ (j) $=$
(k) $\subset$ (l) $\neq$
6. (a) $\varnothing$, {1}, {2}, {1, 2}
(b) $\varnothing$, {pen}, {ink}, {ruler}, {pen, ink}, {pen, ruler}, {ink, ruler}, {pen, ink, ruler}
(c) $\varnothing$,{Singapore}, {Malaysia}, {Singapore, Malaysia}
(d) $\varnothing$, {a}, {e}, {i}, {o}, {a, e}, {a, i}, {a, o}, {e, i}, {e, o}, {i, o}, {a, e, i}, {e, i, o}, {a, e, o}, {a, i, o}, {a, e, i, o}
7. (a) T (b) T
(c) F (d) F
(e) F (f) T
(g) T (h) F
(i) T (j) T
(k) F
8. (a) {20, 40, 60, 80}
(b) {60} (c) {40, 80}
(d) $\varnothing$
10.

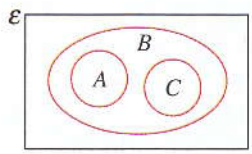


Exercise 10d

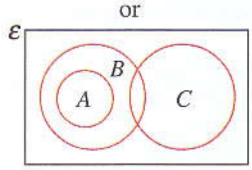
1. (a) {b, c}
(b) $\varnothing$
(c) {m, n, y}
(d) {t, s, c}
(e) {i, r, a, l}
(f) {a, i, o}
2. (a) {1, 2, 3, 4, 5, 7, 9}
(b) {1, 2, 3, 4, 5, 6, 7, 8 9}
(c) {2, 4, 6, 8}
(d) {2, 3, 5, 6, 7, 9}
(e) {2, 4, 6, 8}
3. {durian, mango, pineapple, rambutan, soursop}
{durian, mango}
4. (a) {3, 6, 8, 9, 12}, {6, 9}
(b) {a, b, x, y, m, n, o, p}, $\varnothing$
(c) {monkey, goat, lion, tiger}, {goat}
(d) {a, m, k, y}, $\varnothing$
5. They are the same.
6. They are the same.

Review Exercise 10

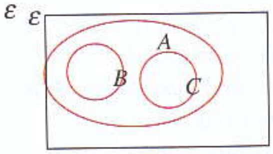
1. (i) {6, 9, 12, 15, 18, 21, 24, 27, 30}
(ii) {4}
(iii) {1, 2, 3, 4, 5}
(iv) $\{5\frac{1}{2}\}$
2. (iv)
3. (ii), (iv), (v)
4. (iv), (v), (vi), (vii), (viii)
5. (i) T (ii) T
(iii) T (iv) F
(v) T (vi) F
(vii) F (viii) F
(ix) F (x) F
(xi) F (xii) T
6. (i) T (ii) T
(iii) F (iv) T
(v) T (vi) T
(vii) T (viii) T
(ix) F
7. (i) 70 (iii) 43
8. (i) {2, 5}
9. (i) {3, 5, 7, 11}
(ii) {1, 2, 3, 5, 7, 9, 11}
(iii) {1, 2, 4, 6, 8, 9, 10, 12}
(iv) {4, 6, 8, 10, 12}
(v) {2}
11.



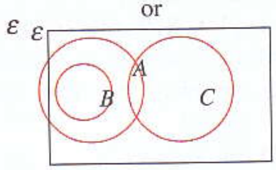
or



12.



or



13. (i) $A = \{28, 40, 52, 64, 76, 88, 100\}$
$B = \{20, 34, 48, 62, 76, 90\}$
(ii) 1

14. (i) A (ii) C
(iii) C

15. (a) 4 (b) 3
(c) 3

18. $C = \{1, 5\}$ or $\{1, 4, 5\}$ or $\{1, 2, 3, 5\}$ etc.
Thus C is not unique.

19. $C = \{$a, c, f$\}$ or $\{$a, c, k$\}$ or $\{$a, c, x, y$\}$ etc.
Thus C is not unique.

Exercise 11a

1. (a) 43 min (b) 15%
Very few executives stay near the office.
2. (a) 90 kg (b) 92 kg
3. (a) 5.4 min (b) $73\frac{1}{3}\%$
About three-quarters of the chidren have short attention span.
4. (a) *XYZ* school
(b) *ABC* school
(c) *ABC* school
5. (a) Kevin (b) Kevin
(d) John. He used more than 80 min/day for 10 days while Kevin used more than 80 min/day for only 4 days.

Exercise 11b

1. (a) 3 (b) 7.7
(c) 14, 16
2. (a) 5 (b) 4
(c) 60 (d) 3
3. 38.1
4. \$35.86
5. (a) 89 (b) 9
(c) 12.2
6. 2
7. 4, 5
8. (a) 246 (b) 47
9. (a) \$8400 (b) \$770
10. 168 cm
11. (a) 27 cm (b) 42 cm
12. 11
13. (a) 6 (b) 13.5
(c) 24.5
14. (a) 57.5 (b) 45
(c) 1.5 (d) 65
15. (a) 12, 13, 14
(b) 7.125, 7, 7
(c) 96.1, 98, 98
(d) 4, 4, 6
16. (a) \$829.63 (b) \$850
(c) \$760
17. 6.4, 6.5, 7
18. (a) 2.58 (b) 2
(c) 2
19. (a) 5 (b) 10
20. (a) 5, 4.67
(b) Peter, No
(c) 3, 4
(d) 2, 6
(e) mode
21. (a) No (b) 7, 7
(c) 8, 6
(d) mode
22. (a) Cheetah
(b) Cheetah
(c) Median score. Because the median score of 87 is higher than the mean score of 75.8. This will give the readers a better information of the team.
23. (a) Carol
(b) All of them
(c) Bruce and Danny, Amy, Carol

Exercise 11c

1. 100.2 g
2. 28.5 cm
3. (a) 172.1 hours
(b) 172.2 hours
(c) 171.8 hours
(d) No; 172.1 hours; How you group your data affects the estimated mean value.
4. 123.52 min
5. 32.65 years old
6. 34 min

Review Questions 11

1. (a) 110 (b) 124
2. (a) 7.2 grams
3. (a) 2 (b) 3 (c) 3.1
4. (c) 20, 24, 3, 3
5. (a) 4, 16 (b) 3, 3
6. (a) 9, 4 (b) 5, 6
7. $x = 26, y = 29$
8. (a) 9 (b) 14 (c) 2
9. (a) $0 \leqslant x \leqslant 11$
(b) 19 (c) 11.4
10. (a) 7 (b) 7 (c) 1
11. (a) 7 (b) 5
12. (b) 130.2 cm (c) 66.7%

Exercise 12a

1. (a) {A, B, C, D, E, F, G}, 7
(b) $\{Y_1, Y_2, Y_3, Y_4, Y_5, G_1, G_2\}$, 7
(c) {1, 2, 3, 4, 5, 6}, 6
(d) {Black, White, Red, Blue, Green}, 5
(e) {100, 101, 102, ..., 999}, 900
(f) {1, 2, 3, 4, 5, 6, 7, 8}, 8
(g) {2, 3, 3, 5}, 4

2. (a) $\frac{1}{8}$ (b) $\frac{1}{2}$
(c) $\frac{1}{2}$ (d) $\frac{1}{4}$
(e) 1 (f) 0

3. (a) $\frac{6}{13}$ (b) $\frac{7}{13}$
(c) 0 (d) $\frac{3}{13}$
(e) $\frac{4}{13}$ (f) $\frac{3}{13}$

4. (a) $\frac{1}{4}$ (b) $\frac{3}{8}$

5. (a) $\frac{9}{80}$ (b) $\frac{1}{10}$
(c) $\frac{13}{80}$ (d) $\frac{17}{80}$
(e) $\frac{1}{5}$

6. (a) $\frac{1}{6}$ (b) $\frac{1}{10}$
(c) 0 (d) 1
(e) $\frac{2}{5}$

7. (a) $\frac{1}{9}$ (b) $\frac{1}{2}$
(c) $\frac{1}{5}$

8. (a) $\frac{2}{11}$ (b) $\frac{4}{11}$
(c) 0 (d) $\frac{7}{11}$

9. (a) $\frac{2}{7}$ (b) 0
(c) $\frac{3}{7}$ (d) $\frac{4}{7}$

10. (a) $\frac{1}{8}$ (b) $\frac{1}{2}$
(c) $\frac{3}{8}$ (d) $\frac{1}{2}$
(e) $\frac{3}{8}$ (f) 0

11. (a) $\frac{1}{5}$ (b) $\frac{3}{5}$
(c) $\frac{1}{5}$ (d) $\frac{2}{5}$

12. $\frac{3}{8}$

13. (a) $\frac{7}{10}$ (b) $\frac{7}{10}$

14. (a) $\frac{2}{5}$ (b) $\frac{1}{15}$

15. (a) $\frac{1}{2}$ (b) $\frac{1}{5}$
(c) $\frac{13}{20}$

16. $\frac{2}{3}$

17. (a) $\frac{1}{2}$ (b) 0
(c) 1 (d) $\frac{3}{10}$
(e) $\frac{1}{5}$ (f) $\frac{7}{10}$

18. (a) $\frac{1}{52}$ (b) $\frac{3}{13}$
(c) 0 (d) $\frac{1}{2}$
(e) $\frac{1}{4}$ (f) $\frac{1}{13}$

19. (a) $\frac{1}{26}$ (b) 0
(c) $\frac{3}{26}$ (d) $\frac{1}{13}$
(e) $\frac{1}{26}$

20. (a) $\frac{1}{27}$ (b) $\frac{13}{27}$
(c) $\frac{13}{54}$ (d) $\frac{4}{27}$
(e) $\frac{1}{54}$ (f) $\frac{2}{27}$

21. (a) $\frac{1}{4}$ (b) $\frac{3}{8}$
(c) $\frac{1}{6}$ (d) $\frac{5}{24}$

22. (a) $\frac{8}{25}$ (b) $\frac{9}{50}$
(c) $\frac{1}{2}$

23. (a) $\frac{3}{16}$ (b) $\frac{3}{16}$
(c) $\frac{3}{4}$

Exercise 12b

1. (a) $\frac{2}{5}$ (b) $\frac{3}{5}$
(c) $\frac{11}{20}$

2. (a) $\frac{5}{12}$ (b) $\frac{7}{12}$
(c) $\frac{1}{4}$ (d) $\frac{3}{4}$

3. {23, 25, 27, 32, 35, 37, 52, 53, 57, 72, 73, 75}
(a) $\frac{1}{4}$ (b) $\frac{1}{4}$
(c) $\frac{1}{3}$ (d) $\frac{1}{3}$

4. {55, 56, 56, 55, 56, 56, 65, 65, 66, 65, 65, 66}
(a) $\frac{1}{3}$ (b) 0
(c) $\frac{5}{6}$

5. (a) (i) $\frac{2}{3}$ (ii) $\frac{1}{3}$
(iii) $\frac{3}{13}$ (iv) $\frac{12}{13}$

6. (a) $\frac{3}{10}$ (b) $\frac{15}{43}$

7. (a) (i) $\frac{2}{5}$ (ii) $\frac{1}{8}$
(b) (i) $\frac{2}{39}$ (ii) $\frac{1}{13}$

8. 7
9. 60
10. 140

11. (a) $\frac{x}{15}$ (b) 10

12. 4

Review Questions 12

1. (a) $\frac{1}{2}$ (b) $\frac{1}{2}$
(c) $\frac{1}{6}$

2 (a) $\frac{7}{22}$ (b) $\frac{9}{22}$
(c) 0 (d) $\frac{3}{11}$

3. (a) 60
(b) (i) $\frac{1}{4}$ (ii) $\frac{5}{18}$
(iii) 0

4. $\frac{5}{17}$

5. (a) $\frac{7}{20}$ (b) $\frac{2}{3}$

6. (a) (i) $\frac{7}{18}$ (ii) 0
(iii) 1
(b) (i) $\frac{7}{30}$ (ii) 0
(iii) $\frac{2}{5}$

7. (a) (i) $\frac{2}{11}$ (ii) $\frac{4}{11}$
(iii) 0
(b) (i) $\frac{1}{10}$ (ii) $\frac{1}{5}$
(iii) $\frac{7}{10}$

8. {2, 5, 7, 25, 27, 52, 57, 72, 75, 257, 275, 527, 572, 725, 752}
(a) $\frac{2}{5}$ (b) $\frac{3}{5}$
(c) $\frac{1}{15}$

9. (a) $\frac{1}{5}$ (b) $\frac{1}{5}$
(c) $\frac{5}{8}$ (d) $\frac{3}{10}$

10. (a) $\frac{31}{110}$ (b) $\frac{21}{55}$

(c) $\frac{101}{110}$ (d) $\frac{9}{110}$

11. (a) $\frac{3}{25}$ (b) $\frac{17}{50}$

(c) $\frac{17}{50}$ (d) 0

(e) 0

12. (a) $\frac{2}{5}$ (b) $\frac{3}{25}$

(d) $\frac{11}{20}$

13. (a) $\frac{183}{242}$ (b) $\frac{20}{121}$

(d) $\frac{2}{121}$

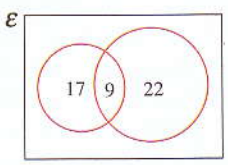
14. (a) (i) $\frac{47}{242}$ (ii) $\frac{15}{121}$

(b) (i) $\frac{31}{313}$ (ii) $\frac{61}{313}$

15. (a) $\frac{1}{2}$ (b) 0

Revision Exercise IV No. 1

1. $k = \frac{c}{x+y}$, $1\frac{1}{4}$
2. (a) 72 km/h
   (b) 20 m/s
3. (a) $(x-1)(x-20)$
   (b) $(x+4)(x+9)$
   (c) $(1+3x)(1-5k)$
4. (a) $\frac{11}{12}$ (b) $\frac{1}{365}$
5.



$n(A) = 26$

6. (a) $\frac{17-2x}{12}$
   (b) $-5a - b$
   (c) $\frac{x^2+2x+3}{x^3}$
   (d) $\frac{a^2-b^2}{ab}$
7. (a) 112 cm³
   (b) 15 cm
   (c) 0.251 cm
8. (a) 11
   (b) 9
   (c) $8\frac{7}{9}$
9. (a) (i) 9.90 cm
   (ii) 141.2 cm³
   (b) 480.0 g
10. 3, –5, –5, 3
    (a) –3.8 (b) –6

Revision Exercise IV No. 2

1. (a) $\frac{x^2+y^2+z^2}{xy}$
   (b) $\frac{7x-8y-6}{6}$
2. (a) 1 (b) 1
   (c) 15 (d) $16\frac{4}{7}$
4. (a) $\frac{1}{2}$ (b) $\frac{23}{90}$
   (c) $\frac{11}{90}$ (d) $\frac{4}{45}$
5. (a) $6x^2 - 7xy - 5y^2$
   (b) $4x^2 - 4xy - 8x - 3y^2 + 12y$
6. 0, 2 or 3
7. (a) 4 (b) $4\frac{1}{6}$
   (c) 40°
8. (a) $\frac{x+y}{xy}$
   (b) $\frac{1}{(x+2)(x+3)}$
   (c) $\frac{5x+y}{(x+y)(x-y)}$ or $\frac{5x+y}{x^2-y^2}$
9. 20, 60
10. 11, 3, 3, 6, 18
    (a) 4.3; 3.6 or –1.6
    (b) $x = 1$

Revision Exercise IV No. 3

1. $1440
2. (a) $b = \frac{cx-ay}{x-y}$, $6\frac{1}{2}$
   (b) 22
3. $12, $10.50
4. (a) $x = 16$ (b) 32
5. (a) R (b) Q
   (c) P (d) Q
6. mean = 13 yr 5 mths
   median = 13 yr 7 mths
7. (a) 31.4 cm² (b) 31 g
8. $8\frac{2}{7}$
9. (a) 13 200 cm³
   (b) 9240 g
10. – 3, – 3, 1
    (a) 0.5 (b) – 4.4
    (c) 2.8 or 1.8

Revision Exercise IV No. 4

1. 10 cm
2. (a) 87.976 cm
   (b) 307.916 cm²
3. (a) $\frac{3}{10}$ (b) $\frac{13}{50}$
   (c) $\frac{1}{10}$
4. 3
6. $x = 4.8$, $y = 13\frac{1}{3}$
7. 52
8. (a) 13 (b) 13
   (c) 12.7
9. 12 cm, 30 cm²
10. 7.9

Revision Exercise IV No. 5

1. 60.7 km/h
2. $168
3. $a = \frac{2s-2ut}{t^2}$, $1\frac{1}{3}$
4. $7, $2.50
5. (a) $\frac{1}{16}$ (b) $\frac{21}{64}$
   (c) $\frac{39}{64}$
6. 4852.672 cm², 311.952 cm
7. 13 520 cm³
8. (a) 11 (b) 12.5
   (c) 13 (d) 15
9. 7.5 days; 1 – 4 days
10. (a) 12 (b) 12
    (c) 37

New Syllabus Mathematics is a series of 4 textbooks written for secondary school students with the ultimate aim to prepare them for the GCE "O" level mathematics examination. The series covers the MOE Syllabus for Secondary Mathematics implemented from 2007.

Special Features

- Exploration activities to promote applications in inter-disciplinary and real life
- Real-life photos at the chapter beginning to provide a thought-provoking introduction
- Graded exercises and challenging problems to cater to different learning abilities
- Real-life situations to demonstrate the relevance of mathematics to everyday life
- Revision exercises to recapitulate previously covered mathematic concepts
- Thinking Time to provide opportunities for creative and individual thinking
- Awareness of problem-solving strategies is systematically enhanced
- Self-assessment to incite active learning and independent thinking
- Your Attention to remind students of common errors
- IT activities to encourage computer-based learning
- Problem Solving Tips to enhance thinking skills